MATHEMATICS
OF
ACCOUNTING AND FINANCE

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PREFACE

This book has been prepared as a manual of business calculation, for use in connection with the problems which continually arise in the course of business as conducted today.

In deciding what subjects to include the authors have been guided throughout by the desire to make the book directly useful in business offices, particularly to persons working in accounting and in the various lines of finance. There are many school texts on commercial arithmetic, but they are necessarily too rudimentary to be of much aid to readers of mature experience. No space has been given here to a consideration of the elementary and fundamental processes of mathematics.

The effort has been, instead, to present material of more advanced nature which has not been generally available. This material falls into three general classes.

The earlier chapters explain in considerable detail a number of short processes and practical suggestions that may be applied in routine computations of any sort. Particular attention has been given to the matter of adequate checks upon calculations.

The central portion of the book treats of the special applications of arithmetical principles and short methods to the problems of individual lines of business.

In the final chapters an attempt has been made to explain, in simple terms, convenient ways of using logarithmic and actuarial methods in the solution of business problems relating to compound interest, investments, annuities, bond discount and premium, effective bond rates, leaseholds and depreciation.

. . . . . .

During the last months of his life, Mr. Walton devoted most of his strength to this book, and it is a source of profound personal

regret that he did not live to see its publication. Few men were his equal as an accountant, a teacher and a writer. None surpassed him as a friend.

H. A. FINNEY.

EVANSTON, ILL.
August 15, 1921

CONTENTS

Chapter		Page
I	Short Methods and Practical Suggestions	3
	Short Methods of Computation Balancing an Account Combined Addition and Subtraction Adding a Part of a Column Deducting Several Subtrahends from one Minuend Table with Net Decrease Complements Short Methods of Multiplication Multiplying by 11 Multiplying by 111 Tabulating Multiples of a Multiplier Tabulating Multiples of a Divisor Division by Use of Reciprocals	
II	Fractions and Proof Figures	16
	Addition and Subtraction of Fractions Cross Multiplication Equivalent Common and Decimal Fractions Proof Figures	
III	Arithmetical Progression	22
	Elements in Arithmetical Progression Analysis of Simple Progression Computing Total Simple Interest C. P. A. Problem	
IV	Average	29
	Utility of Average Simple Average Moving Average Progressive Average Periodic Average Weighted Average	
V	Averaging Accounts	42
	Settling an Account Calculating Interest Items of Varying Amounts Focal Date Rules Applied	

CHAPTER PAGE

Reducing Days to Months
Compound Average
Example of Compound Average
Another Illustration

VI PERCENTAGE 50

Percentage
Terms Used in Percentage
Fundamental Processes
Percentage of Increase and Decrease
Some Applications of Percentage in Business
Per Cent of Goods Sold—Various Manufacturers
Monthly Sales Compared on a Percentage Basis
Sales for the Week Ending December 18, 1920
Comparison of Sales by Departments
Individual Sales Compared with Average
Individual Sales Compared with Maximum
Apportionment
Division of Profits
Distribution of Factory Overhead
Gross Profit Method of Approximating Inventory
Analysis of Statements
Percentage Analysis to Determine Causes of Variation in Profits

VII EQUATION IN THE SOLUTION OF PROBLEMS 66

Solving Equations
Illustration 1
Illustration 2
Illustration 3
Illustration 4

VIII TRADE AND CASH DISCOUNT 76

Trade Discount
Cumulative Trade Discounts
Methods of Finding Net Price
Cash Discount
Discount as a Protection Against Loss
Cash Discount Regarded as an Expense

IX TURNOVER 83

Indefinite Meaning of "Turnover"
Normal Inventories Necessary
Different Bases of Comparison
Working Capital as Basis of Turnover
Definition of Working Capital
Need of Exact Definitions

CHAPTER PAGE

X PARTNERSHIPS 89

Division of Profits
Liquidation of Partnerships
Periodical Distributions
Reducing Capitals to Profit and Loss Ratio

XI THE CLEARING HOUSE 98

Principle of the Clearing House
Debits and Credits with Clearing House
Clearing House Transactions
Manager's Sheet
Economy of System
Application of Principle Extended

XII BUILDING AND LOAN ASSOCIATION 103

General Characteristics of Building and Loan Associations
Terminating Plan
Practicability of Plan
Serial Plan
Distribution by Partnership Plan
Distribution by Dexter's Rule
Withdrawal of Shares
Sources of Income
Premiums
Individual Plan
Dayton or Ohio Plan

XIII GOOD-WILL AND CONSOLIDATION 116

Purchasing a Business with Stock
Allocation of Net Earnings
Good-Will
Appraising Good-Will—Year's Purchase Method
Capitalizing Gross Income
Issue of Two Classes of Stock

XIV FOREIGN EXCHANGE 123

Conversion of Foreign Coinage
Reverse Conversion
Dealing in Foreign Exchange
Average Date of Current Account
Conversion of Foreign Branch Accounts
Reconciliation of Accounts

XV LOGARITHMS 135

Use
Multiplication
Division

CHAPTER PAGE

Calculating Powers
Roots
Nature of Logarithms
The Characteristic and the Mantissa
Tables of Logarithms
Characteristics of Logarithms of Numbers Between 1 and 10
Use of the Characteristic in Pointing off Results
Logarithms of Numbers Smaller Than 1
Use of Negative Characteristics in Pointing off Answers
Computing with Logarithms having Negative Characteristics
Determining Mantissa by Interpolation
Determining Numbers by Interpolation

XVI SIMPLE AND COMPOUND INTEREST 154

Simple Interest—Methods of Calculating
Rates Other Than 6%
365-day Basis
Partial Payments
Compound Interest
Symbols
Amount of Principal
Frequency of Compounding
Determining the Amount
Determining Interest
Determining Present Worth
Determining the Compound Discount
Summary

XVII ANNUITIES 171

Definition of Annuities
Symbols
Amount of an Annuity
Sinking Fund Contribution
Present Worth of an Annuity
Rent of an Annuity
Equal Periodical Payments on Principal and Interest
Annuities Due
To Find the Amount of an Annuity Due
Sinking Fund
Required Annual Contribution
Present Worth of an Annuity Due
Rents

XVIII LOGARITHMS IN COMPOUND INTEREST AND ANNUITY COMPUTATIONS 190

Calculating Compound Interest and Annuities by Logarithms
To Find the Compound Interest on 1
To Find the Principal
To Find the Rate
To Find the Time

CHAPTER PAGE

To Find the Present Value of 1
To Find the Compound Discount on 1
To Find the Amount of an Annuity
To Find the Amount of Sinking Fund Contributions
To Find the Present Worth of an Annuity

XIX BOND DISCOUNT AND PREMIUM 194

Bonds Purchased Below and Above Par
Discount
Scientific Method of Amortization
Income Rates
Bond Premium
Computing the Premium and the Price
First Method
Second Method
Computing the Discount and the Price
First Method
Purchases at Intermediate Date
Serial Bonds

XX LEASEHOLDS 217

Commuted Rents

XXI DEPRECIATION METHODS 221

Annual Depreciation

APPENDIX A—VALUES OF FOREIGN COINS 231

B—LOGARITHMS OF NUMBERS 233

C—COMPOUND INTEREST AND OTHER COMPUTATIONS . 254

Mathematics of Accounting and Finance

CHAPTER I

SHORT METHODS AND PRACTICAL SUGGESTIONS

Short Methods of Computation

Since the work of an accountant necessarily involves a great deal of computation, it is desirable that he be familiar with certain labor-saving devices. Those which are presented in this chapter have been selected because of their simplicity and because an accountant has occasion to apply them so frequently as to make their use habitual. No attempt is made here to describe all the innumerable "short methods" which, however ingenious, are difficult to remember and rarely available.

Balancing an Account

In striking a balance in an account, the proper method for determining and inserting the balance is to add the larger column and enter its total in both columns; then to add the smaller column and insert each figure of the balance necessary to produce the total. The following example is given to illustrate the method.

Debits	Credits
$1,846.22	$ 126.13
2,913.68	248.71
4,327.11	1,635.48
	2,319.11
	4,757.58
$9,087.01	$9,087.01

The balance of $4,757.58 in this account is found and entered in the following way. First, the larger side of the account, which happens to be the debit side, is added for a total of $9,087.01 and the amount is placed on both sides; then the items on the credit

side are added and the figure in each digit column necessary to give the figure in the corresponding column of the total is inserted in the place reserved for the balance.

Thus, adding the first column, $3 + 1 + 8 + 1$, gives 13, to which 8 is added to make 21 and produce the 1 in the total. The 8 is inserted in the balance as shown, while the 2 of 21 is carried. The second column of digits and the 2 carried forward are added to 15, and to this a 5 is added to make 20 and give 0 in the total. The 5 is entered in the same column of the balance and the 2 of 20 is carried. In the same way the third column is added to 30, 7 is inserted in the balance to make 37, and the 3 of 37 is carried. The fourth column is added to 13, 5 is inserted to produce 18, and 1 is carried. The fifth column is added to 13, 7 is inserted to make 20, and 2 is carried. The last column requires a 4 to make 9 in the total, and the entire balance is found to be $4,757.58.

Combined Addition and Subtraction

It often happens that columns of figures are given in which are included both positive and negative values, that is, plus and minus numbers. It may be necessary, for example, to compare the figures of two years to show the increase or decrease of certain items and to determine the net increase or decrease for the second year. If the figures are written in ink, the minus quantities may be inserted in red. In a newspaper or book the minus numbers are usually printed in italics or indicated by a star.

Two methods may be followed in adding a column of such figures. The first method is the ordinary way of adding the positive numbers and subtracting from their sum the sum of the negative numbers. The other method is to add the numbers continuously, the negative digits being in each case added negatively, that is subtracted. Thus, in adding the third column of the following table, which is a comparison of the manufacturing costs of 1916 with those of 1915, the procedure is as explained below:

Account	1915	1916	Increase or Decrease
Material used	$12,638.13	$14,228.64	$1,590.51
Direct labor	16,469.42	19,672.18	3,202.76
Indirect labor	5,827.59	4,713.92	*1,113.67*
Other manufacturing expense	7,962.28	8,242.37	280.09
	$42,897.42	$46,857.11	$3,959.69

The net increase of $3,959.69 for all four items is found by adding the increases and decreases in the following manner. The first column of digits on the right is added thus, $1 + 6 = 7, 7 - 7 = 0, 0 + 9 = 9$. The second column is added thus, $5 + 7 = 12$, $12 - 6 = 6$. In the third column 3 is subtracted from 2 to make -1, and the 1 is subtracted from 10 borrowed from the next column. The addition is therefore, $2 - 3 + 10 = 9$. In the fourth column the borrowed 10 is represented by -1, and the column adds thus, $-1 + 9 - 1 + 8 = 15$. The 1 of 15 is carried, as in ordinary addition, to the next column, which added is, $1 + 5 + 2 - 1 + 2 = 9$. The final column adds, $1 + 3 - 1 = 3$.

In practice, items of the same value but of opposite signs may be offset at once. Thus, a glance at the cents column in the foregoing example shows that 1 and 6 offset -7, and the remaining 9 can, therefore, be put down at once in the result.

Adding a Part of a Column

This manner of addition is useful when it is desired to find the sum of all but the last few numbers of a long column whose total is known. For instance, a page may have an amount on each of 40 lines, and the addition of the entire column has been verified. It is now necessary to ascertain the total of the first 35 lines, as in the illustration below, in which these first 35 lines are omitted to save space.

36th line	$ 2,382.46
37th "	1,896.28
38th "	1,237.61
39th "	2,068.42
40th "	1,923.16
Total of 40 amounts on page	$82,642.57

The total of the items to and including the 35th may be found by adding the last five items on a memorandum paper and deducting their sum from the total of the page. A much quicker and more workmanlike procedure, however, is to subtract them by addition as explained in the previous section. Adding the first column of digits, $6 + 8 + 1 + 2 + 6$ gives 23. As 4 is required to make the 7 in the grand total, 4 must be the last digit in the footing at the 35th line. Carrying 2 and proceeding in the same way with the second and the other columns, the required footing is found to be $73,134.64. The entire operation involves no more writing than the insertion of the figures under the 35th line.

Deducting Several Subtrahends from One Minuend

It is sometimes necessary to subtract several amounts from one amount, as, for example, to make various deductions from total sales. If there is room enough the various items may be listed in an inside column, their total inserted under gross sales, and the difference carried out. If only one column is possible, the subtraction can be made by addition, as shown in the following table, in which the items between the lines are deductions from the total sales, and the remainder is the amount of the net sales.

Total sales	$132,629.14
Less:	
Cash discount	$ 1,436.27
Freight	2,389.16
Breakages	948.63
Sundry allowances	769.42
Net sales	$127,085.66

In performing the operation it should be remembered that the digits of the remainder are always those necessary in each column to make the digit of the same column in the total.

Table with Net Decrease

At times the amounts to be deducted are greater than the amount from which they are to be deducted. The table below is an example of this. It represents a comparison between two years to show the increase or decrease of business in each of seven departments and the total net result of the whole business. The table might have been arranged in four columns, one for the figures of 1920, one for the figures of 1921, one for the increases, and one for the decreases; or the departments might have been listed in order, with the changes appearing in one column but the decreases indicated by red ink or italics. In the table as given the departments showing an increase are listed first, and those showing a decrease are listed underneath.

Dept.	1920	1921	Increase or Decrease	
1	$168,242.19	$174,629.46	Inc.	$ 6,387.27
3	192,429.36	195,716.83	"	3,287.47
4	156,283.27	160,142.76	"	3,859.49
7	84,619.43	86,223.62	"	1,604.19
Total	$601,574.25	$616,712.67	Tot. Inc.	$15,138.42
2	$165,328.46	$158,693.82	Dec.	$ 6,634.64
5	212,642.88	204,976.38	"	7,666.50
6	198,731.54	189,218.23	"	9,513.31
Total	$576,702.88	$552,888.43	Tot. Dec.	$23,814.45
Grand total	$1,178,277.13	$1,169,601.10	Net Dec.	$8,676.03

The net decrease in the foregoing table is the sum of the decreases less the sum of the increases. The total of the increases is, therefore, added negatively to the total of the decreases, that is, deducted digit by digit.

Complements

The arithmetical complement of a number is the difference between that number and the next higher power of 10. Thus the complement of 7 is 3, or the difference between 10 and 7, and the complement of 81 is 19, or the difference between 100 and 81.

If instead of subtracting a number less than 10, its complement is added, the result is 10 larger than the result of the subtraction. Thus 8 subtracted from 15 gives 7; while the complement of 8, which is 2, added to 15, gives 17. Hence to subtract digits, in adding a column of numbers negatively, their complements may be added to the other digits and the sum reduced by ten times the number of complements added.

The following table represents an addition of positive and negative quantities:

929
248
− 1,086
− 74
3,126
− 174
306
3,275

In the unit column of this table, 9 + 8 + 4 (complement of 6) + 6 (complement of 4) + 6 + 6 (complement of 4) + 6 = 45, and 45 − 30 = 15, 30 being subtracted because three complements have been added. The 5 of the 15 is entered in the footing and the 1 is carried. In the tens column 1 (carried) + 2 + 4 + 2 (complement of 8) + 3 (complement of 7) + 2 + 3 (complement of 7) = 17. The 7 is entered in the footing, and since three complements were added the 1 is dropped and − 2 carried to the hundreds column. In the hundreds column 8 (complement of the − 2 carried) + 9 + 2 + 1 + 9 (complement of 1) + 3 = 32, and 32 − 20 (there being only two complements) = 12. The 2 is entered and the 1 is carried. In the thousands column 1 + 9

(complement of 1) + 3 = 13, and 13 − 10 (there being only one complement) = 3, which is entered in the footing. The result of the addition is, therefore, 3,275.

Short Methods of Multiplication

To multiply numbers ending in zeros, both of which are integers, the significant figures of the numbers are multiplied and to the product are annexed as many zeros as there are final zeros in both the multiplicand and the multiplier. Thus, 3,400 times 1,200 equals 34 times 12, or 408, with four zeros annexed, or 4,080,000.

When one of the two numbers multiplied is a decimal fraction the significant figures in the two numbers are multiplied and the decimal point in the product is moved as many places to the right as there are final zeros in the integer. This may necessitate annexing zeros. In the example .486×300, .486 is first multiplied by 3, to make 1.458. As there are two zeros in the multiplier, the decimal point in this product is moved two places to the right, making it 145.8.

To multiply by 9, 99, or any number that contains no other figure than 9, as many zeros are annexed to the multiplicand as the multiplier has 9's, and from the result the multiplicand is deducted. Thus, in multiplying 293 by 99, two zeros are annexed to 293, making 29,300, and from this 293 is deducted. The result is 29,007.

Multiplying by 11

To multiply an amount by 11 it is not necessary to put down the figures, but only the result, which is secured by addition. The procedure is as follows:

Beginning with the right-hand or unit figure in the multiplicand, add to each digit the digit next to the right of it. Put the last digit of this sum down in the result, carrying the other digit of the sum if there is one. Continue this until each figure in the

multiplicand has been used twice. The method is illustrated in the following example:

Illustration

Multiply 1,342 by 11

SOLUTION:

1,342

Product 14,762

The first digit, 2, at the right of the multiplicand, has no figure to the right of it. Therefore nothing is added and it remains	2
The second digit, 4, is added to the first digit, 2, making	6
The third digit, 3, is added to the second digit, 4, making	7
The fourth digit, 1, is added to the third digit, 3, making	4
Since the fourth digit has been used only once it is necessary to use it a second time, thus,	1
The product is, therefore,	14,762

An analysis of the ordinary method of multiplication shows that the procedure is exactly the same in both cases. Looking at the example when multiplied in the usual way, it is seen that,

1342
11
1342
1342
14,762

As in the above explanation, the product is formed by putting down 2 alone, then adding 4 to 2, 3 to 4, 1 to 3 and repeating the 1. It is, however, unnecessary to write out these figures, as the whole operation can be performed mentally and the product put down at once.

Multiplying by 111

To multiply by 111 the same procedure is followed as when the multiplier is 11, except that each figure in the multiplicand is added to the two figures to the right, and is, therefore, used three times, as shown in the following illustration:

Illustration

Multiply 1,342 by 111

SOLUTION:

1,342

Product 148,962

The first digit in the multiplicand is put down alone, thus,	2
The second digit, 4, is added to the first digit, 2, making	6
The third digit, 3, is added to the second digit, 4, and the first digit, 2, making	9
The fourth digit, 1, is added to the third digit, 3, and the second digit, 4, making	8
The fourth digit, 1, is added to the third digit, 3, making	4
The fourth digit, 1, is put down alone, thus,	1
	148,962

If the same method is applied in multiplying 1,796 by 111, it will be found necessary to carry over 1, 2 and 1, in adding the digits.

In multiplying by any number that consists of a series of 1's each digit in the multiplicand is used as many times in the way explained above as there are digits in the multiplier; twice when the multiplier is 11, three times when it is 111, four times when it is 1,111, and so on.

Tabulating Multiples of a Multiplier

When a number containing several digits is to be used repeatedly as a multiplier, it saves time and promotes accuracy to make a table of its multiples. The best way of compiling the

table is to determine each multiple by adding the number to the preceding multiple. The addition is, of course, made on the table itself and not on a separate sheet of paper. The table is carried out to ten multiples in order to prove the work, as the tenth line must be the same as the first with one zero annexed. This is proof only in case the multiples are built up by addition.

A table of this sort constructed for 683,947 as a multiplier is the following:

Multiplier	Multiple
1	683,947
2	1,367,894
3	2,051,841
4	2,735,788
5	3,419,735
6	4,103,682
7	4,787,629
8	5,471,576
9	6,155,523
10	6,839,470

Supposing it is desired to multiply 683,947 by 3,469, the solution is as follows:

683,947 multiplied by 9 =		6155523
" " " 6 =		4103682
" " " 4 =		2735788
" " " 3 =		2051841
Product		2372612143

Tabulating Multiples of a Divisor

When a number containing several figures is to be used repeatedly as a divisor, a table of multiples of the divisor may be prepared, to show at a glance how many times the divisor is contained in successive remainders and do away with the necessity of performing the multiplication. This method may also save considerable labor by reason of the fact that instead of writing

down the multiples to make each subtraction, they may be written on a card and the particular multiple to be deducted placed alongside the amount from which it is to be subtracted. The subtraction is made from the card across to the amount, and only the remainder is set down.

The directions for the use of this method of division may be given as follows:

Placing the card at the left of the dividend, find by inspection which multiple is to be used; slide the card until that multiple is on a line with the dividend; deduct the multiple from the corresponding first figures of the dividend, putting the remainder under the dividend figures; then bring down the next figure and proceed in the same way with the new amount to be divided. The following example, in which 872,976,654 is divided by 683,947, illustrates this method.

Card of Multiples (as slid for each division)		Dividend	Quotient
1	683,947	872976654	1
2	1,367,894	1890296	2
7	4,787,629	5224025	7
6	4,103,682	4363964	6
3	2,051,841	2602820	3
8	5,471,576	5509790	8
		38214	

Quotient is 1,276.38, and the remainder 382.14.

Division by Use of Reciprocals

Two numbers are reciprocal when their product is 1. For instance, 2 and .5 are reciprocal numbers, because $2 \times .5 = 1$. It is evident from this that the reciprocal of a number is found by dividing 1 by the number. The quotient obtained by dividing by a divisor is the same as the product obtained by multiplying by the reciprocal of the divisor. For example, $12 \div 2$ and $12 \times .5$ are each equal to 6.

Repeated division by the same divisor can be made easier, particularly if an adding machine is available, by determining the reciprocal of the number, tabulating the multiples of the reciprocal, and proceeding as in multiplication. In dividing, for example, 872,976,654 by 683,947, the first step is to divide 1 by the divisor in order to obtain its reciprocal, which is .0000014621. The multiples of the reciprocal are then tabulated, as in the table below. The zeros in the reciprocal are not repeated in the table, as it is only necessary to remember the number of places to be pointed off in the product, ten in the present case. Having the multipliers, the procedure thenceforth is precisely as in multiplication, as shown in the second of the subjoined tables.

MULTIPLES OF RECIPROCAL

1	14,621
2	29,242
3	43,863
4	58,484
5	73,105
6	87,726
7	102,347
8	116,968
9	131,589
10	146,210

14621
872976654
58484
73105
87726
87726
102347
131589
29242
102347
116968
1276379165813 4

Pointing off ten decimals, the result is 1,276.38

Since the reciprocal is approximate only, the last figures of the product must be ignored, as they do not show the exact remainder.

The work may be shortened considerably by beginning at the left and ignoring all multiples and portions thereof beyond the first, second, or third place to the right of the last place desired in the product, as follows:

	14621
	872976654
8	116968
7	102347
2	29242
9	131589
7	10235
6	877
6	88
5	7
4	1
	127637917 or 1,276.38

CHAPTER II

FRACTIONS AND PROOF FIGURES

Addition and Subtraction of Fractions

Fractions having the same numerator may be added by multiplying the sum of the denominators by the common numerator to obtain the numerator of the result, and by multiplying together the denominators of the fractions to obtain the denominator of the result which is then reduced to its simplest form. To add 1/9 and 1/7 by this method the procedure is as follows:

$$9 + 7 = 16;\ 16 \times 1 = 16, \text{ the numerator of sum}$$
$$9 \times 7 = 63, \text{ the denominator of sum}$$
$$\text{Sum} = 16/63$$

The solution of 3/11 + 3/16 is as follows:

$$11 + 16 = 27;\ 27 \times 3 = 81, \text{ the numerator of sum}$$
$$11 \times 16 = 176, \text{ the denominator of sum}$$
$$\text{Sum} = 81/176$$

To subtract fractions when the numerators are the same, the difference between the denominators is first multiplied by the common numerator. This gives the numerator of the result. Then the denominators are multiplied to obtain the denominator of the result, which is reduced to its simplest terms. This method is illustrated in the solution of the following subtraction, 3/11 − 3/16:

$$16 - 11 = 5;\ 5 \times 3 = 15, \text{ the numerator of the difference}$$
$$16 \times 11 = 176, \text{ the denominator of the difference}$$
$$\text{Difference} = 15/176$$

Cross Multiplication

Two fractions may be added by the method of cross multiplication. The numerator of each is multiplied by the denominator

of the other, and the two products are added to form the numerator of the result. The denominators of the fractions are multiplied to form the denominator of the result, which is then reduced to its simplest terms. This method is illustrated in the following example, in which 5/8 and 7/9 are added.

$5 \times 9 = 45$
$7 \times 8 = 56$
101, numerator of the sum
$8 \times 9 = 72$, denominator of the sum
Sum = $101/72$ or $1\ 29/72$

Cross multiplication may also be used in subtracting fractions. The numerator of each fraction is multiplied by the denominator of the other, and the smaller product is subtracted from the larger to form the numerator of the result. The two denominators are multiplied to form the denominator of the result. Reduction to the simplest form then follows. In the following example 5/8 is subtracted from 7/9 by this method:

$7 \times 8 = 56$
$5 \times 9 = 45$
11, numerator of the difference
$9 \times 8 = 72$, denominator of the difference
Difference = $11/72$

The approximate product of two mixed numbers may be found by multiplying first the integers, then each integer by the other fraction to the nearest unit, and finally by adding the three products. The following illustration in which 17 1/5 is multiplied by 14 1/3 shows the application of the method:

$14 \times 17 = 238$
$1/5$ of $14 = 3$ (nearest unit)
$1/3$ of $17 = 6$ " "
Approximate product 247

The exact product of this multiplication is 246 8/15.

If the multiplication of the fractions is carried out to at least one decimal place, the result is more exact, as the following shows:

$$
\begin{aligned}
14 \times 17 &= 238 \\
1/5 \text{ of } 14 &= 2.8 \\
1/3 \text{ of } 17 &= 5.7 \\
\text{Result} \quad & 246.5
\end{aligned}
$$

Equivalent Common and Decimal Fractions

The following is a list of the most frequently used decimal fractions and their equivalent common fractions.

Decimal Fractions	Equivalent Common Fractions
.5	1/2
.33 1/3	1/3
.66 2/3	2/3
.25	1/4
.75	3/4
.16 2/3	1/6
.83 1/3	5/6
.125	1/8
.375	3/8
.625	5/8
.875	7/8
.08 1/3	1/12
.41 2/3	5/12
.58 1/3	7/12
.91 2/3	11/12
.0625	1/16
.1875	3/16
.3125	5/16
.4375	7/16
.5625	9/16
.6875	11/16
.8125	13/16
.9375	15/16

These figures can be used in a variety of ways. One of the most obvious is to substitute the common fraction for its equivalent decimal, as it is usually easier to multiply or divide by the

common fraction than by the decimal. Thus, if it is desired to find 12 1/2% of a number, it is far easier to divide the number by 8 than to multiply it by the decimal .125.

This substitution of a common fraction for the equivalent decimal is simple enough when the numerator of the common fraction is 1. When, however, the numerator is greater than 1, it is necessary to split up the decimal into such parts as will permit the substitution of equivalent common fractions with a numerator of 1. The number taken is multiplied by each of these parts and the products are added to find the result. The fewer the parts into which the decimal is split, the simpler is the operation. The application of the rule is illustrated in the following problem.

Example

How much will 3,648 yards of cloth cost at 43 3/4 cents per yard?
SOLUTION:
The answer can be found in three ways:

1. By multiplying 3,648 by $.4375.
2. By substituting for $.4375 its equivalent, 7/16 of $1, and multiplying 3,648 by the latter.
3. By splitting up the common fraction of 7/16 into 1/4, 1/8, and 1/16, or the decimal of .4375 into .25, .12 1/2, and .06 1/4, and substituting the equivalent fractions, and then multiplying 3,648 by each of these fractions and adding the products, as shown below:

At $1.00 per yard the price would be	*$3,648.00*
At 1/4 or .25 per yard the price would be	912.00
At 1/8 or .12 1/2 (1/2 of above)	456.00
At 1/16 or .06 1/4 " " "	228.00
At 7/16 or .43 3/4 the price is	$1,596.00

The operation can be performed much more quickly if it is recognized that 7/16 is the same as 8/16 minus 1/16, that is, 1/2 minus 1/8 of 1/2, and the calculation is made as shown below:

1/2 of $3,648 =	$1,824.00
1/8 of 1,824 =	228.00
7/16 of $3,648 =	$1,596.00

It is always advisable to prove all computations when it does not involve too much work. This is easily done in most cases where a fraction can be split up. Thus, in the above example it has been found that 7/16 of $3,648 is $1,596. If the figure for 1/16 is added to instead of subtracted from that for 8/16 the result is 9/16, and the calculation of $1,596 can be proved as follows:

8/16 minus 1/16 is	7/16 or	$1,596.00
8/16 plus 1/16 is	9/16 or	2,052.00
	16/16 or original amount	$3,648.00

Proof Figures

There are two systems of proof figures for testing the accuracy of mathematical computations. One is based on the casting out of 9's and the other on the casting out of 11's. Many bookkeepers use one or the other system to check all their work. Both systems are used in the illustration below to check an addition, the figure at the right of each amount being the excess over the 9's or 11's in the amount, or the remainder left after the digits are added and the largest multiple of 9 or 11 is subtracted.

	Excess over 9's			Excess over 11's	
365,592	3		365,592	7	
286,548	6		286,548	9	
64,320	6		64,320	3	
94,094	8	Total 23, or an	94,094	0	Total 19, or an
810,554	5	excess of 5	810,554	8	excess of 8

To cast out 9's from a number, the digits are added ignoring 9's, zeros and combinations of digits which add to 9. From the

sum thus obtained the largest multiple of 9 contained therein is subtracted. Thus, the digits in the first number in the above example are added as follows, beginning at left: Ignore the 3 and 6 because they add to 9; 5 + 5 = 10; ignore the 9; 10 + 2 = 12. 12 − 9 = 3, the excess.

To cast out 11's begin with the first figure at the right and add to it the third, fifth and so on, and take the excess of their sum over 11's. Then add together the second, fourth, sixth figures, etc., and find the excess of their sum over 11's. Subtract this excess from the first excess and the result is the check number. If the first excess is smaller than the second, add 11 to it before subtracting. Thus, in the first number of the foregoing example, 2 + 5 + 6 = 13, or an excess of 2 over 11. Similarly, 9 + 5 + 3 = 17, or an excess of 6 over 11. Adding 11 to the 2 and subtracting 6 gives 7 as the check number.

Addition is proved if the excess over 9's or 11's in the sum of the individual check numbers is the same as the check number of the total of the numbers added, as shown in the preceding example. This is not an absolute proof, as an error of 9 or 11 or multiples thereof may be made.

Multiplication is proved by multiplying the check figure of the multiplicand by that of the multiplier. The excess over 9's or 11's in the result should be the check figure of the product, as is seen from the following:

Numbers	Excess over 9's	Excess over 11's
4,621	4	1
3,274	7	7
15,129,154	28 or 1	7

CHAPTER III

ARITHMETICAL PROGRESSION

Elements in an Arithmetical Progression

An arithmetical progression is a series of numbers increasing or decreasing by a common difference. The numbers in the series are called the terms; the first and last terms are called the extremes, and the intermediate terms the means. An increasing or ascending series is formed by adding the common difference to each preceding term. For example, 7, 12, 17, 22, 27, is an ascending series with a common difference of 5. A decreasing or descending series is formed by subtracting the common difference from each preceding term. Thus, 26, 23, 20, 17, 14, is a descending series with a common difference of 3.

There are five elements in an arithmetical progression, which in the formulas to be presently derived are represented by the following symbols:

First term.....f
Last term.....l
Common difference.....d
Number of terms.....n
Sum of series.....s

When any three of these elements are known, the other two can be computed.

Analysis of Simple Progression

The following is an example of a short progression:

1st term	3
2nd "	6
3d "	9
4th "	12
5th "	15
6th or last term	18

The common difference here is 3, the number of terms is 6, and the sum of the terms is 63.

It will be seen that, although the number of terms is 6, the common difference is added only five times. This explains why in Cases 1 and 2 considered below, 1 is subtracted, and why in Case 3, 1 is added. It will also be seen that the series consists of a number of pairs, as follows:

1st and last terms make a pair the sum of which is 21
2nd " 5th " " " " " " " " " " 21
3d " 4th " " " " " " " " " " 21

The sum of series is 63

It is evident from this that the sum of a series is the sum of the first and last terms multiplied by half of the number of terms. It is also evident that since the sum of each pair of terms is 21, the average single term is 10½, which multiplied by 6, or the number of terms, gives 63, or the sum of the series.

If the number of terms is uneven, the number of pairs includes a half pair. Thus, if the series given above is extended to seven terms, the last term is 21, and the sum of the terms is 84, which is 3½ times 24, the sum of the first and last terms.

In the following four cases the derivation of the formulas for computing the various elements of an arithmetical progression is explained.

Case 1. Given the first term, common difference and the number of terms, to find the last term.

If the series is ascending, the common difference must be added as many times, less one, as there are terms in the series. Hence the formula is:

$$l = f + (n - 1)\, d$$

Example

The first term is 9, the common difference is 3, and the number of terms is 5. Find the last term.

SOLUTION:

$$\begin{aligned} l &= 9 + (5 - 1)\,3 \\ &= 9 + 12 \\ &= 21 \end{aligned}$$

If the series is descending, the common difference must be deducted as many times, less one, as there are terms in the series. Hence the formula is:

$$l = f - (n - 1)\,d$$

Example

The first term is 21, the common difference is 3, and the number of terms is 5. Find the last term.

SOLUTION:

$$\begin{aligned} l &= 21 - (5 - 1)\,3 \\ &= 21 - 12 \\ &= 9 \end{aligned}$$

Case 2. Given the extremes and the number of terms, to find the common difference.

The number of common differences is one less than the number of terms; and the sum of the common differences is the difference between the extremes. Hence in an ascending series the formula for finding the common difference is:

$$d = \frac{l - f}{n - 1}$$

and in a descending series it is:

$$d = \frac{f - l}{n - 1}$$

Example

The first term is 9, the last term is 21, and the number of terms is 5. Find the common difference.

SOLUTION:

$$d = \frac{21 - 9}{5 - 1}$$
$$= \frac{12}{4}$$
$$= 3$$

Case 3. Given the extremes and the common difference, to find the number of the terms.

The difference between the extremes is the sum of the common differences, and the number of the common differences is one less than the number of terms. Hence in an ascending series the formula for calculating the number of terms is:

$$n = \frac{l - f}{d} + 1$$

and in a descending series it is:

$$n = \frac{f - l}{d} + 1$$

Example

The first term is 9, the last term is 21, and the common difference is 3. Find the number of terms:

SOLUTION:

$$n = \frac{21 - 9}{3} + 1$$
$$= \frac{12}{3} + 1$$
$$= 4 + 1$$
$$= 5$$

Case 4. If the extremes and the number of terms are given, the sum of the terms is found by the following formula:

$$s = \frac{f + l}{2} \times n$$

or by the following:

$$s = (f + l) \times \frac{n}{2}$$

Example

The extremes are 9 and 21, and the number of terms is 5. Find the sum of the terms.

SOLUTION:

$$s = \frac{9 + 21}{2} \times 5 = 75 \text{ or } s = (9 + 21) \times \frac{5}{2} = 75$$

Computing Total Simple Interest

The most important applications of arithmetical progression with which accountants are concerned fall under Case 4. They are made in computing the total simple interest on a principal which constantly increases or decreases from period to period by a common difference.

For example, a $1,000,000 bond issue bearing interest at 5% is to be repaid in forty equal annual instalments, and it is necessary to compute the total interest to be paid during that period. Since 1/40 of the loan is to be paid annually, the principal during the last year will be 1/40 of $1,000,000 or $25,000. The interest for the first year will be 5% of $1,000,000, or $50,000, which is the first term of an arithmetical progression containing forty terms; the interest for the last year will be 5% of $25,000, or $1,250, which is the last term of the progression. Since

$$s = \frac{f + l}{2} \times n$$

the total interest, which is s, will be

$$\frac{\$50{,}000 + \$1{,}250}{2} \times 40, \text{ or } \$1{,}025{,}000$$

C. P. A. Problem

The application of the principle of arithmetical progression to the computation of interest on a given principal for an increasing or decreasing series of time periods, may be illustrated by a problem similar to one given in an Illinois C. P. A. examination. The problem is as follows:

Problem

Upon the death of a retired business man in June, 1910, a will is found conveying real and personal property aggregating $300,000 to the widow, who is his second wife, for her life, and upon her death to four children in equal shares. It is discovered after his death that his first wife had left to her two children, Henry and Emma, $20,000, consisting of securities for $10,000 bearing 6% interest, and uninvested cash of $10,000. The father had regularly collected the semi-annual interest on the investment, but there was no evidence as to his disposition of the cash portion of the bequest. Exactly ten years elapsed between the death of his first wife and his own death, so that he had collected twenty items of interest, the last one just before he died. Henry and Emma were of age at the time of their father's death, and had never been informed of their legacy.

Prepare a statement showing what would accrue to each of the four children at the death of the widow, who died immediately after her husband, including the amounts to which Henry and Emma would be entitled on account of their mother's estate. Exclude and do not consider any accrued income of the estate unexpended.

In Illinois the legal rate of interest on undisclosed debts is 5%.

SOLUTION: The proceeds of the property that belonged to his first wife constitute a trust fund belonging to the two children, Henry and Emma; the total of this fund on June 1, 1900, comprises:

Open account for cash collected by decedent, June, 1890	$10,000.00
Interest thereon at 5% (legal rate on undisclosed debt) for ten years	5,000.00
Principal of securities	10,000.00
Interest collected on securities	6,000.00

Each collected coupon increased the undisclosed debt of the father as guardian. The first coupon was collected

9 1/2 years or 19 half years before the father's death. The twentieth coupon had just been collected. The computation of the interest on these coupons can be accomplished thus:

First $300 earns interest at 5% for 9 1/2 years	$142.50
Last $300 " " " " " 0 "	0
Sum of extremes	$142.50
Number of terms (coupons)	20

Applying the formula:

$$s = \frac{f + l}{2} \times n$$

or

$$s = (f + l) \times \frac{n}{2}$$

The total interest = 142.50 × 10 or	1,425.00
Total due Henry and Emma from mother's estate	$32,425.00

The division of the estate would be as follows:

Total real and personal property	$300,000.00
Amount due Henry and Emma from mother's estate	32,425.00
Balance divided among four children	$267,575.00
Of which one-fourth is	$ 66,893.75

The allowance of $1,425 for interest on the coupons collected must not be confused with compound interest. The item of $1,425 is interest on actual cash received by the father, for which he did not account. How he came into possession of the cash is immaterial.

CHAPTER IV

AVERAGE

Utility of Average

The principle of average may be used for determining probabilities, for comparing numbers with a standard or with each other, or for the purpose of simplifying calculations by using an average instead of a number of related values. Where it is possible to collate statistics covering extensive and varied observations, it is possible to determine an average which may be assumed, from the law of averages, to be standard. Reliance can then be placed on the probability that other similar cases, though individually at great variance from the average, will in the aggregate closely approximate the average. Mortality tables, for instance, are averages determined by exhaustive investigation. While they do not determine probabilities for individuals, they do determine probabilities for large groups of individuals.

When the principle of average is utilized, the basis of comparison may be a simple average, a moving average, a progressive average, a periodic average, or a weighted average as the case may be. The choice of a base depends on the information desired and, in the case of weighted average, on the necessity imposed by the facts themselves.

Simple Average

The process of determining a simple average consists merely of adding the units to be averaged and dividing the sum by the number of units. If the sum of the units is known, the process requires division only.

Suppose the daily sales for a week are as follows:

Monday, $3,560.35; Tuesday, $3,115.95; Wednesday, $2,946.86; Thursday, $2,868.79; Friday, $3,269.87; Saturday, $3,896.43. Total sales, $19,658.25.

The daily average of the sales for the week is found by dividing the total of $19,658.25 by the number of days taken, or 6, which gives a result of $3,276.37½.

It must be remembered in figuring average that the divisor is the number of units in the dividend, and not necessarily the number of items added to obtain the dividend. For example, suppose it is desired to determine the average contribution made by a number of persons to a benevolent fund, where

2	men	each	contributed	$500,	totaling $1,000.00
4	"	"	"	250,	" 1,000.00
10	"	"	"	100,	" 1,000.00
16	"	"	"	50,	" 800.00
18	"	"	"	25,	" 450.00
50	"	"	"	a total of	$4,250.00

The number of items added here to find the dividend of $4,250 is only 5, but the number of units to be used as the divisor is 50, making the average contribution equal to $4,250 divided by 50 or $85.

Moving Average

When it is desired to compare a series of numbers relating to units of time of uniform duration and of uninterrupted sequence, a moving average of a number of these units may be used as a basis of comparison. This kind of average is determined by taking a simple average of the numbers to serve as the starting point for the moving series, and dropping after the lapse of each time unit the first number of the series and adding the number of the next time unit in order to form a new series and obtain a new average.

Assume, for example, the following conditions:

1. The values averaged are monthly sales approximated to the nearest $1,000.
2. The time units are months.
3. The number of units is 12, making up one year.
4. The first month of each series is dropped, as a new month is included in the next series.

In an example of this sort each month's sales can be compared with any one of the twelve averages in the calculation of which the month's sales are included. For instance, the sales of December, 1914, can be compared with the twelve monthly averages for the years ending with each month from December, 1914, and to November, 1915, inclusive.

Some of the possible comparisons in which the moving average may be used are illustrated in the table below. Their value depends on the nature of the business whose figures are used. The most significant fact shown in the illustration is the almost uninterrupted increase in the moving average, although the monthly sales show wide variation due to the seasonal nature of the business. The decline in the moving average during the last months of 1914 and the first months of 1915, shows the effects of the war.

Following is an explanation of the methods by which the figures in the various columns are determined:

Moving average:

Total sales for 12 months of 1913	335
Moving average including December, 1913, 335 ÷ 12	27. 9
335 − 20 (January, 1913) + 53 (January, 1914)	368
Moving average including January, 1914, 368 ÷ 12	30.66

Increase or decrease in moving average—1 month:*

Moving average including January, 1914	30.7
" " " December, 1913	27.9
Increase	2.8

More or less than annual average:*

December, 1913, sales	27
Moving average including December	27.9
Decrease	.9*

January, 1914, sales	53
Moving average including January	30.7
Increase	22.3

Increase in moving average—1 year:

Moving average including December, 1914	35.4
" " " December, 1913	27.9
Increase	7.5

Increase in moving average—2 years:

Moving average including January, 1916	48
" " " January, 1914	30.7
Increase	17.3

MOVING AVERAGE

1913	Sales (000 omitted)	Moving average	More or less* than annual average
Jan.	20		
Feb.	17		
Mar.	21		
Apr.	17		
May	26		
June	40		
July	28		
Aug.	21		
Sept.	36		
Oct.	55		
Nov.	27		
Dec.	27	27.9	.9*

MOVING AVERAGE—*Continued*

1914	Sales (000 omitted)	Moving average	Increase or decrease* in moving average—1 month	More or less* than annual average	Increase in moving average—1 year
Jan.	53	30.7	2.8	22.3	
Feb.	47	33.2	2.5	13.8	
Mar.	47	35.3	2.1	11.7	
Apr.	45	37.7	2.4	7.3	
May	32	38.2	.5	6.2*	
June	32	37.5	.7*	5.5*	
July	24	37.2	.3*	13.2*	
Aug.	17	36.8	.4*	19.8*	
Sept.	39	37.1	.3	1.9	
Oct.	42	36.0	1.1*	6.0	
Nov.	20	35.4	.6*	15.4*	
Dec.	27	35.4	.0	8.4*	7.5

1915	Sales (000 omited)	Moving average	Increase or decrease* in moving average—1 month	More or less* than annual average	Increase or decrease* in moving average—1 year
Jan.	45	34.8	.6*	10.2	4.1
Feb.	43	34.4	.4*	8.6	1.2
Mar.	39	33.8	.6*	5.2	1.5*
Apr.	38	33.2	.6*	4.8	4.5*
May	32	33.2	.0	1.2*	5.0*
June	25	32.6	.6*	7.6*	4.9*
July	27	32.8	.2*	5.8*	4.4*
Aug.	32	34.1	1.3	2.1*	2.7*
Sept.	49	34.9	.8	14.1	2.2*
Oct.	85	38.5	3.6	46.5	2.5
Nov.	67	42.4	3.9	24.6	7.0
Dec.	47	44.1	1.7	2.9	8.7

MOVING AVERAGE—*Continued*

1916	Sales (000 omitted)	Moving average	Increase or decrease* in moving average— 1 month	More or less* than annual average	Increase in moving average — 1 year	Increase in moving average —2 years
Jan.	92	48.0	3.9	44.0	13.2	17.3
Feb.	97	52.5	4.5	44.5	18.1	19.3
Mar.	83	56.2	3.7	26.8	22.4	20.9
Apr.	44	56.7	.5	12.7*	23.5	19.0
May	33	56.8	.1	23.8*	23.6	18.6
June	42	58.2	1.4	16.2*	25.6	20.7
July	38	59.1	.9	21.1*	26.3	21.9
Aug.	55	61.0	1.9	6.0*	26.9	24.2
Sept.	80	63.6	2.6	16.4	28.7	26.5
Oct.	97	64.6	1.0	32.4	26.1	28.6
Nov.	84	66.0	1.4	18.0	23.6	30.6
Dec.	58	66.9	.9	8.9*	22.8	31.5

Progressive Average

Progressive average is cumulative, a new unit being added to form each successive dividend, and the divisor being constantly increased in an arithmetical progression of 1. In the table given below the figures used are those of the preceding illustration. The first average is that of the first two months' sales; the second, of the first three months' sales, etc.

The table shows the continuous growth of the business; but the differences between the successive progressive averages are not so significant as the differences between the successive moving averages, because the former are borne down by the smaller sales of the first months and years. Moreover, an increase of sales in an early month increases the progressive average for that month to a greater extent than the same increase in sales in a later month will increase the progessive average of that month, because the total sales in the latter case are divided by a larger number of months. The last column of the table indicates the months in which the sales run above or below the average. It

will be noted that the current month is not included in the average used as a base. If it were included, any increase or decrease in the month's sales would affect the standard as well as the month compared.

Following is an explanation of the methods of determining the figures appearing in the various columns:

Progressive average:

February line	(20 + 17) ÷ 2	18.5
March line	(20 + 17 + 21) ÷ 3	19.3

Increase or decrease in progressive average:*

Average for first three months	19.3
" " " two months	18.5
Increase	.8

More or less than progressive average:*

Sales of March, 1913	21
Progressive average for prior months	18.5
Increase	2.5

PROGRESSIVE AVERAGE

1913	SALES	PROGRESSIVE AVERAGE	INCREASE OR DECREASE* IN PROGRESSIVE AVERAGE	MORE OR LESS* THAN PROGRESSIVE AVERAGE
January	20			
February	17	18.5		
March	21	19.3	.8	2.5
April	17	18.8	.5*	2.3*
May	26	20.2	1.4	7.2
June	40	23.5	3.3	19.8
July	28	24.1	.6	4.5
August	21	23.8	.3*	3.1*
September	36	25.1	1.3	12.2
October	55	28.1	3.0	29.9
November	27	28.0	.1*	1.1*
December	27	27.9	.1*	1.0*

PROGRESSIVE AVERAGE—*Continued*

1914	SALES	PROGRESSIVE AVERAGE	INCREASE OR DECREASE* IN PROGRESSIVE AVERAGE	MORE OR LESS* THAN PROGRESSIVE AVERAGE
January	53	29.8	1.9	25.1
February	47	31.1	1.3	17.2
March	47	32.1	1.0	15.9
April	45	32.9	.8	12.9
May	32	32.9	.0	.9*
June	32	32.8	.1*	.9*
July	24	32.4	.4*	8.4*
August	17	31.6	.8*	15.4*
September	39	32.0	.4	7.4
October	42	32.4	.4	10.0
November	20	31.9	.5*	12.4*
December	27	31.7	.2*	4.9*
1915				
January	45	32.2	.5	13.3
February	43	32.7	.5	10.8
March	39	32.9	.2	6.3
April	38	33.0	.1	5.1
May	32	33.0	.0	1.0*
June	25	32.7	.3*	8.0*
July	27	32.5	.2*	5.7*
August	32	32.5	.0	.5*
September	49	33.0	.5	16.5
October	85	34.6	1.6	52.0
November	67	35.5	.9	32.4
December	47	35.8	.3	11.5
1916				
January	92	37.3	1.5	56.2
February	97	38.9	1.6	59.7
March	83	40.0	1.1	44.1
April	44	40.1	.1	4.0
May	33	40.0	.1*	7.1*
June	42	40.0	.0	2.0
July	38	40.0	.0	2.0*

PROGRESSIVE AVERAGE—*Continued*

1916—*Cont.*	SALES	PROGRESSIVE AVERAGE	INCREASE OR DECREASE* IN PROGRESSIVE AVERAGE	MORE OR LESS* THAN PROGRESSIVE AVERAGE
August	55	40.3	.3	15.0
September	80	41.2	.9	39.7
October	97	42.4	1.2	55.8
November	84	43.3	.9	41.6
December	58	43.6	.3	14.7

Periodic Average

In order to show the variation in the volume of business between seasons, periodic average may be utilized, as in the illustration below, in which a simple average is taken of the figures for the same month in the years 1913–1916 inclusive:

MONTH	1913	1914	1915	1916	TOTAL	AVERAGE
January	20	53	45	92	210	52.5
February	17	47	43	97	204	51.0
March	21	47	39	83	190	47.5
April	17	45	38	44	144	36.0
May	26	32	32	33	123	30.75
June	40	32	25	42	139	34.75
July	28	24	27	38	117	29.25
August	21	17	32	55	125	31.25
September	36	39	49	80	204	51.0
October	55	42	85	97	279	69.75
November	27	20	67	84	198	49.50
December	27	27	47	58	159	39.75

The principle of progressive average may be utilized in connection with the principle of periodic average, as indicated in the following table:

YEAR	JANUARY	PROGRESSIVE AVERAGE	MORE OR LESS* THAN PRECEDING PROGRESSIVE AVERAGE
1913	20		
1914	53	36.5	
1915	45	39.3	8.5
1916	92	52.5	52.7

Each progressive average in the foregoing illustration is also a periodic average. This organization of the numbers and averages makes possible a comparison of the sales of each month with the average sales of the same month in all preceding years. The figures in the final column show the difference between the January sales of the year and the progressive average of the January sales of the preceding years. Thus the figure 45 for 1915 is 8.5 greater than the 36.5 progressive average for January, 1913, and 1914.

Weighted Average

When the values entering into the computation of an average differ in two or more particulars, a simple average is impossible. Take the following case, for example:

3	men	earn	$5.00	per	day
5	"	"	6.00	"	"
4	"	"	7.00	"	"

In computing the average daily wage of these twelve men, the fact must be recognized that the wage payments differ in two particulars:

1. The daily wage
2. The number of men receiving each wage

The daily average is found by dividing the aggregate of their daily wages by their number, as shown below:

Men	Wage	Product
3	$5	$15
5	6	30
4	7	28
12		$73

$\$73 \div 12 = \$6\frac{1}{12}$, the average daily wage

This example serves to illustrate the principle that each value must be weighted by multiplying it by the number of units to

which the value is applicable. Additional illustrations will make the principle clearer.

Illustration 1

What is the average rate of interest earned on the following investments made for one year?

$100 at 7%
$5,000 at 6%
$15,000 at 5%

SOLUTION:

PRINCIPAL	RATE	PRODUCT
$ 100	7%	$ 7.00
5,000	6%	300.00
15,000	5%	750.00
$20,100		$1,057.00

1,057 ÷ 20,100 = 5.258%, the average rate of interest

Illustration 2

What is the average life of a plant, the various fixed assets of which have the following costs and estimated lives?

CLASS OF ASSET	COST	LIFE
A	$ 3,000.00	5 years
B	15,000.00	10 "
C	35,000.00	20 "

SOLUTION:

CLASS	COST	LIFE	DEPRECIATION RATE	ANNUAL DEPRECIATION
A	$ 3,000.00	5 years	20%	$ 600.00
B	15,000.00	10 "	10%	1,500.00
C	35,000.00	20 "	5%	1,750.00
	$53,000.00			$3,850.00

53,000 ÷ 3,850 = 13.76+ %, the average life

Illustration 3

What is the average per cent of increase in the cost of manufacture under the following conditions: Where one pound of each item is used in the manufacture of each completed article, and the cost of each item has increased by the per cent shown in the last column of the first of the subjoined tables?

Material	Cost per Pound 1917	Per cent Increase 1918
A	$ 4	200
B	5	10
C	8	6¼
D	15	10
E	18	20

Solution 1:

Material	No. Pounds	Cost per Pound 1917	Per cent Increase 1918	Increase 1918
A	1	$ 4	200	$8.00
B	1	5	10	.50
C	1	8	6¼	.50
D	1	15	10	1.50
E	1	18	20	3.60
		$50		$14.10

14.10 ÷ 50 = 28.2%, average increase

Solution 2:

Material	Cost per Pound 1917	Per cent of Total	Per cent Increase 1918	Weighted Per cent 1918
A	$ 4	8	200	16.0
B	5	10	10	1.0
C	8	16	6¼	1.0
D	15	30	10	3.0
E	18	36	20	7.2
	$50	100		28.2

Illustration 4

The same conditions are taken here as in the preceding illustration, except that the items composing the finished article are each of a different weight, as indicated in the second column of the following two solutions:

SOLUTION 1:

MATERIAL	NO. POUNDS	% TOTAL	COST PER POUND 1917	WEIGHTED PRODUCT 1917	% INCREASE 1918	WEIGHTED % 1918
A	2	10	$ 4	$ 40	200	80
B	8	40	5	200	10	20
C	4	20	8	160	6¼	10
D	4	20	15	300	10	30
E	2	10	18	180	20	36
	20	100		$880		176

176 ÷ 880 = 20%, weighted average per cent

SOLUTION 2:

MATERIAL	NO. POUNDS	COST PER POUND 1917	TOTAL COST 1917	% INCREASE 1918	INCREASE 1918
A	2	$ 4	$ 8	200	$16.00
B	8	5	40	10	4.00
C	4	8	32	6¼	2.00
D	4	15	60	10	6.00
E	2	18	36	20	7.20
			$176		$35.20

35.20 ÷ 176 = 20%, weighted average per cent

CHAPTER V

AVERAGING ACCOUNTS

Settling an Account

The object of averaging an account is to determine a single date, known as the average date, on which the account may be settled with fairness to both debtor and creditor.

As a simple illustration of how an account is averaged, suppose B's account with A, to whom he is indebted, is as follows:

March 1—60 days	$1,000
March 31—60 days	1,000

The first item of this account is due on April 30 and the second on May 30. B may pay each amount at its maturity, or the entire $2,000 at the average maturity, which is May 15. He can make an equitable settlement on this average date because the time he gains in deferring payment of the first $1,000 for fifteen days is exactly offset by the time he loses in paying the second $1,000 the same number of days in advance of the due date.

It may be desirable to compute the average date for the purpose of dating or determining the maturity of a note or other document given in settlement of an account. For instance, B might cover his account by giving A a single non-interest bearing note for $2,000, due May 15, instead of two non-interest bearing notes of $1,000 each, one due April 30, and the other due May 30.

Calculating Interest

The average date may also be desired for the purpose of computing interest on the balance instead of the individual items of an account. For instance, if A should settle the account on June 14 by a single payment of $2,000, equity would require that he pay interest, say, at 6% as follows:

On $1,000 from April 30 to June 14—45 days	$ 7.50
On $1,000 from May 30 to June 14—15 days	2.50
Total	$10.00

The interest might, however, be computed on the entire balance of $2,000 for a period of thirty days from the average date, May 15 to June 14. Its amount in this case would also be $10.

Assume that instead of paying cash, B, on April 14 gave A a note for $2,000 due in two months. The maturity of this note would be June 14, or thirty days after the average date. An equitable settlement would require the addition of $10 interest to the face of the note, which would make it $2,010.

Items of Varying Amounts

In the foregoing illustrations the amounts are the same and only the varying number of days has had to be considered in arriving at the average date. If, however, the amounts are not the same, they must also be considered. Suppose, for example, that B's account with A was as follows:

March 1—60 days	$2,000
" 31—60 days	1,000

The $2,000 item is due April 30 and the $1,000 item is due May 30. In this case the average date of maturity for the total of $3,000 would not be May 15 as in the previous example, since the payment of $2,000 fifteen days after maturity would not be offset by the payment of $1,000 fifteen days before maturity. It would be May 10, because $2,000 paid ten days after maturity would be counterbalanced by $1,000 paid twenty days before maturity. It is evident from this that in averaging accounts due consideration must be given to amounts as well as to dates.

The dates to be used in averaging an account are those at which the items may be assumed to have a cash value equivalent to the amount at which they appear in the accounts. Thus, sales on cash terms, being due on the day of sale, take their invoice

dates in averaging; sales with credit terms take the dates on which the invoices are due; returns and allowances take the dates when the invoices to which they apply are due; an interest-bearing note takes the date of the note since the face is the cash value at that date; a non-interest bearing note is not worth its face until due and hence its maturity is used in the calculation of the average date.

Focal Date

The average dates in the preceding illustrations were determined by inspection. When an account is not so simple the computation of its average date requires a method involving the principle of weighted average and the selection of a basic date for calculating the time of each item. This basic date is called the *focal date* and the one most advantageously employed is the last day of the month preceding the earliest date used in averaging.

Take, for example, the following account:

Date of Transaction	Terms of Payment	Date Used in Averaging	Amount
March 1	60 days	April 30	$2,000.00
" 31	60 "	May 30	1,000.00

The first date to be used in averaging the account is April 30. Hence March 31, the last day of the preceding month, is selected for the focal date. The subsequent steps in the process of averaging the account may be enumerated as follows:

1. Assume that each item is paid on the focal date.
2. Determine the number of days each item would be prepaid if it were paid on the focal date.
3. Multiply its amount by this number of days.
4. Add the products thus obtained.
5. Divide this sum by the total of the items; the quotient represents the number of days the focal date precedes the average date.

Rules Applied

The application of these rules to the foregoing account is as follows:

1. The two items are assumed to be paid on March 31, the focal date taken.
2. The first item is therefore assumed to be prepaid 30 days, and the second 60 days.
3. Paying $2,000 30 days before it is due is equivalent to paying $1, 60,000 (2,000 × 30) days before it is due; and paying $1,000 60 days before it is due is equivalent to paying $1, 60,000 (1,000 × 60) days before it is due.
4. Hence the two assumed prepayments are equivalent to a prepayment of $1 by 120,000 days.
5. One dollar prepaid 120,000 days is equivalent to $3,000 prepaid 1/3,000 of 120,000 days, or 40 days. The average date is therefore 40 days forward from March 31. Forty days are taken as the equivalent of 1 month and 10 days, making the average date May 10.

The computation of the average date is shown in tabular form as follows:

Date of Transaction	Terms of Payment	Date in Average	Time from Focal to Maturity Date	Amount	Product
March 1	60 days	April 30	30 days	$2,000	$60,000
March 31	60 "	May 30	60 "	1,000	60,000
				$3,000	$120,000

Dividing the sum of the products by the sum of the amounts (120,000 ÷ 3,000) gives 40, or the number of days the average date follows the focal date.

Reducing Days to Months

When the time between the focal date and the date of any of the items in the account is more than one month, the reduction of this time to days may be avoided by the method outlined below:

1. Express the time in months and days.
2. Multiply the amount of each item by the number of months to obtain a product of months; and by the number of days to obtain a product of days.
3. Add the products of months; also the products of days.
4. Reduce the sum of the products of months to days by multiplying by 30.
5. To the product of days thus obtained, add the product of days previously obtained.
6. Divide this sum by the balance of the account to find the time in days between the focal and the average dates.
7. Reduce this time to months and days on the basis of thirty days to a month.

This method is illustrated in the following example, in which the focal date taken is February 28, being the last day of the month preceding the earliest date in the average, which is March 3.

Date of Transaction	Terms of Payment	Date in Average	Time			Product	
			Mos.	Days	Amount	Mos.	Days
March 3	cash	March 3	0	3	$250	$ 0	$ 750
March 18	1 month	April 18	1	18	500	500	9,000
April 10	30 days	May 10	2	10	200	400	2,000
May 8	cash	May 8	2	8	400	800	3,200
					$1,350	$1,700	$14,950
						30 × 1,700 =	51,000
							$65,950

65,950 ÷ 1,350 = 48 115/135, or 49 days
49 days = 1 month and 19 days

One month and nineteen days forward from February 28, the focal date, is April 19, which is the average date.

This computation is based on the assumption that there are 30 days in each month and 360 days in a year. Strictly speaking, however, the item due May 10 runs for seventy-one days instead

of seventy, and the item due May 8 runs for sixty-nine days instead of sixty-eight. The result, however, would not be materially modified if the exact number of days were taken. The other method is, therefore, sufficiently accurate for all ordinary commercial transactions.

Compound Average

The average date of an account is computed by simple average, when the account contains either debit or credit items, but not both. When both debits and credits are included, the average date is found by means of compound average, which involves the following steps:

1. Determine the products of months and days for the debits and credits.
2. Determine the difference between the sum of the debit products and the sum of the credit products.
3. Divide this difference by the balance of the account.
4. If the difference of the products is on the same side of the account as its balance, the average date is forward from the focal date; but if the difference of the products and the balance of the account are on different sides, the average date is backward from the focal date. This latter condition rarely occurs if the focal date selected is prior to the dates of the items in the account.

Example of Compound Average

In illustrating compound averaging the following account is taken:

	Debit			Credit	
June	1	$500	July	5 Note (2 mo. without int.)	$500
"	20, 1 mo.	400	"	10 Returns (Inv. June 20)	50
July	10	600	August	1 Cash	300
August	5	500			

As the July 5th note on the credit side of the account is non-interest bearing, it does not have a cash value of $500 until its maturity on September 5. It therefore takes this date in the average. The July 10th credit, being an offset to the debit of June 20, takes the same maturity date in the average as the debit item, or July 20th. As June 1 is the earliest date on which any of the items in the account has a cash value equal to the face of the item, the most convenient focal date is May 31.

The sum of the products of months and days for the debit items of the account is found as follows:

Date of Transaction	Terms of Payment	Date in Average	Time Mos.	Time Days	Amount	Products Mos.	Products Days
June 1		June 1	0	1	$500	$ 0	$ 500
" 20	1 month	July 20	1	20	400	400	8,000
July 10		July 10	1	10	600	600	6,000
August 5		August 5	2	5	500	1,000	2,500
Total debits					$2,000	$2,000	$17,000
					30 × 2,000 =		60,000
Sum of the debit products							$77,000

The sum of the products of months and days for the credit items is found by the following computation.

Date of Transaction	Terms of Payment	Date in Average	Time Mos.	Time Days	Amount	Products Mos.	Products Days
July 5	2 months	Sept. 5	3	5	$500	$1,500	$ 2,500
July 10	offset						
June 20	Dr.	July 20	1	20	50	50	1,000
August 1		August 1	2	1	300	600	300
Total credits					$850	$2,150	$ 3,800
					2,150 × 30 =		64,500
Sum of the credit products							$68,300

The account has a debit balance of $1,150 and the sum of its debit products exceeds the sum of its credit products by 8,700.

The average date is forward from May 31, the focal date, as many days as the number of times 1,150 is contained in 8,700, or approximately 8. June 8 is, therefore, the focal date. The account could be averaged by deducting the $50 credit from the June 20th debit of $400 and dealing only with the net debit of $350.

Another Illustration

To illustrate the conditions under which the average date is backward from the focal date, assume that the July 5th non-interest bearing note was due in six months. All of the debit items are due on or before August 5, but since the creditor would have to wait until January 5 for the $500 payable on the note, which in the meantime would earn no interest, the balance of the account should carry an early average maturity. The sum of the debit products of days would be computed as in the preceding example and would total 77,000, while the sum of the credit products of days would be computed as follows:

Date of Transaction	Terms of Payment	Date in Average	Time Mos.	Time Days	Amount	Product Mos.	Product Days
July 5	6 months	January 5	7	5	$500	$3,500	$ 2,500
July 10	offset	July 20	1	20	50	50	1,000
August 1		August 1	2	1	300	600	300
Total credits					$850	$4,150	$ 3,800
					30 × 4,150 =		124,500
Sum of the credit products							$128,300

The balance of the account would still be a debit of $1,150, but the difference of the products of days would now be on the credit side, and would amount to 51,300. The average date would, therefore, be backward from June 1, the focal date, by forty-five days, or the number of times $1,150 is contained in 51,300. Counting thirty days to the month, the average date would be one month and fifteen days backward from the focal date and would, therefore, be April 15.

CHAPTER VI

PERCENTAGE

Percentage

Percentage is a method of computing by hundredths. The symbol % means per cent or hundredths. A rate per cent is equivalent to a common fraction the numerator of which is expressed and the denominator of which is indicated by the symbol % as being 100. Thus the same facts may be stated in the form of a common fraction, a decimal fraction or a per cent. The following are equivalent:

Common Fractions	Decimal Fractions	%
17/100	.17	17
1¼	1.25	125
7½	7.50	750
12/4	3.00	300

Terms Used in Percentage

Base. The number of which a given per cent is to be taken is called the base.

Rate. The per cent of the base to be taken is called the rate.

Percentage. The result obtained by taking a certain per cent of the base is called the percentage.

Fundamental Processes

All mathematical computations involving percentage may be grouped under three headings:

1. To find a given per cent of a number; that is, to find the percentage.

   Rule: Multiply the base by the rate

   Example: $60 × 20% = $12

   Base × Rate = Percentage

2. To find what per cent one number is of another; that is, to find the rate.

Rule: Divide the percentage by the base

Example: $12 ÷ $60 = 20%

Percentage ÷ Base = Rate

3. To find a number when a certain per cent of it is known; that is, to find the base.

Rule: Divide the percentage by the rate

Example: $12 ÷ 20% = $60

Percentage ÷ Rate = Base

Percentage of Increase and Decrease

Percentage is frequently employed to compare numbers and to show how much larger or smaller one number is than the other. No new mathematical principles are involved in such computations, as may be shown by the following illustrations:

1. *Percentage of Increase.* The sales of a certain business in May, 1919, were $16,000 while the sales in May, 1920, were $18,000.

The smaller number is taken as the base; the difference between the two numbers is the percentage of increase. Then

2,000 ÷ 16,000 = 12½%

Percentage of increase ÷ Base = Per cent of increase

2. *Percentage of Decrease.* The profits of a business for the year 1919 were $20,000 while the profits for 1920 were $17,000.

The larger number is taken as the base; the difference between the two numbers is the percentage of decrease. Then

3,000 ÷ 20,000 = 15%

Percentage of decrease ÷ Base = Per cent of decrease

Some Applications of Percentage in Business

Comparisons

Business statistics may be tabulated and compared on a percentage basis to determine the relative effectiveness, desirability

or productivity of similar factors. The illustrations given in this chapter are intended to be suggestive but not exhaustive.

The following tabulation shows the number of units of a certain commodity purchased from various manufacturers during a year, the number sold, and the per cent sold. A comparison of the per cents indicates the relative salability of the goods purchased from the various manufacturers.

PER CENT OF GOODS SOLD—VARIOUS MANUFACTURERS

MANUFACTURER	NUMBER PURCHASED	NUMBER SOLD	PER CENT SOLD
Walker & Co.	368	293	79.6
White & Dudley	493	316	64.1
Davis Mfg. Co.	639	582	91.1
Barton-Walsh	523	416	79.5

The following tabulation compares the sales of various salesmen during a month. Each man's sales (as a percentage) divided by the total sales (as a base) produces a rate which measures his portion of the total.

MONTHLY SALES COMPARED ON A PERCENTAGE BASIS

NAME	SALES	PER CENT OF TOTAL
Arthur Bradley	$ 1,264.90	7.8
J. B. Henderson	1,913.52	11.8
Fred Bates	1,732.69	10.6
Arthur Dutton	2,213.72	13.6
J. L. Weston	1,963.45	12.1
Carter Doane	1,627.32	10.0
Walter S. Waite	1,692.18	10.4
Frank Chesley	2,138.45	13.1
Harold Peters	1,728.46	10.6
Total	$16,274.69	100.0

The statistics may be so arranged as to obtain two percentage analyses, as illustrated in the following tabulation which shows

what per cent of the sales of the week was made by each salesman, and what per cent was made each day.

SALES FOR THE WEEK ENDING DECEMBER 18, 1920

DAYS	SMITH	BROWN	JONES	WHITE	DAILY TOTALS	PER CENTS
Monday	$362.50	$562.83	$862.94	$126.39	$1,914.66	17.6
Tuesday	415.75	475.92	732.83	143.62	1,768.12	16.3
Wednesday	396.21	415.60	769.42	129.38	1,710.61	15.8
Thursday	472.96	516.29	640.20	145.17	1,774.62	16.4
Friday	387.29	429.36	721.32	96.27	1,634.24	15.1
Saturday	493.89	562.64	816.25	163.92	2,036.70	18.8
Salesmen's totals	$2,528.60	$2,962.64	$4,542.96	$804.75	$10,838.95	
Per cents	23.3	27.3	41.9	7.4		

The following tabulation illustrates the use of percentage of increase and decrease as a means of comparing the sales of each department of a store on the corresponding days of two years.

COMPARISON OF SALES BY DEPARTMENTS

DEPARTMENT	SALES—THURSDAY DECEMBER 15, 1919	SALES—THURSDAY DECEMBER 16, 1920	INCREASE DECREASE*	% INCREASE—DECREASE*
A	$ 826.95	$ 914.32	$ 87.37	10.57
B	1,034.78	1,231.64	196.86	19.02
C	1,237.62	1,196.14	41.48*	3.35*
D	2,643.80	2,843.27	199.47	7.54
E	1,413.80	1,376.29	37.51*	2.65*
F	962.40	1,235.96	273.56	28.42
G	2,642.16	2,927.92	285.76	10.81
H	1,964.39	2,129.80	165.41	8.42
I	1,636.48	1,596.27	40.21*	2.46*
J	1,213.42	1,723.96	510.54	42.07
Total	$15,575.80	$17,175.57	$1,599.77	10.27

The average of a number of quantities may be accepted as the basis of comparison, the relation of each quantity to the average

being shown in terms of per cents. The figures in the preceding tabulation of "Monthly Sales Compared on a Percentage Basis" are used for the following illustration:

INDIVIDUAL SALES COMPARED WITH AVERAGE

NAME	SALES	PER CENT OF AVERAGE
Arthur Bradley	$1,264.90	69.95
J. B. Henderson	1,913.52	105.82
Fred Bates	1,732.69	95.82
Arthur Dutton	2,213.72	122.42
J. L. Weston	1,963.45	108.58
Carter Doane	1,627.32	89.99
Walter S. Waite	1,692.18	93.58
Frank Chesley	2,138.45	118.26
Harold Peters	1,728.46	95.58
Average	$1,808.30	100.00

Or the maximum may be accepted as the basis of comparison, the relation of all quantities to the maximum being shown in terms of per cents. Using the same statistics for an illustration:

INDIVIDUAL SALES COMPARED WITH MAXIMUM

NAME	SALES	PER CENT OF MAXIMUM
Arthur Bradley	$1,264.90	57.14
J. B. Henderson	1,913.52	86.44
Fred Bates	1,732.69	78.27
Arthur Dutton	2,213.72	100.00
J. L. Weston	1,963.45	88.69
Carter Doane	1,627.32	73.51
Walter S. Waite	1,692.18	76.44
Frank Chesley	2 138.45	96.60
Harold Peters	1,728.46	78.08

The following tabulation is suggestive of the use which may be made of percentage in comparing quantities with two or more similar quantities and with the average thereof. In this case the average is a progressive one.

COMPARISON OF SALES OF SUCCESSIVE YEARS

YEAR	SALES	INC. OR DEC.* FROM PRECEDING YEAR		INC. OR DEC.* FROM FIRST YEAR		PROGRESSIVE AVERAGE PRIOR YEARS	INC. OR DEC.* FROM PROGRESSIVE AVERAGE	
		Amount	%	Amount	%		Amount	%
1917	$200,000							
1918	238,000	$38,000	19.00	$38,000	19.00			
1919	190,000	48,000*	20.17*	10,000*	5.00*	$219,000	$29,000*	13.24*
1920	285,000	95,000	50.00	85,000	42.50	209,333	75,667	36.15

Apportionment

When a quantity is to be divided or partitioned, the basis of the partition may be expressed in rates per cent. The partition is then accomplished by applying to the base a number of rates, the total of which is 100%. The division of partnership profits is a familiar illustration.

DIVISION OF PROFITS

PARTNERS	P. & L. RATIO	PROFITS
A	20%	$3,200.00
B	35%	5,600.00
C	45%	7,200.00
Total	100%	$16,000.00

Or the apportionment may be accomplished by applying the same rate to a number of bases to obtain the desired percentages. In this case the rate is computed by dividing the total percentage by the total of the bases. The distribution of factory overhead is illustrative.

DISTRIBUTION OF FACTORY OVERHEAD

(Direct Labor Cost Basis)

Total direct labor, all departments..... $5,295.00 (base)
Total factory overhead..... 3,460.00 (percentage)
Then $3,460 \div 5,295 = 65.34+\%$ (rate)

Since the rate is approximate only, the distribution will not be exact; a remainder of 24 cents will be undistributed.

Process or Department	Direct Labor	Proportion of Overhead
1	$2,140.00	$1,398.28
2	1,965.00	1,283.93
3	1,190.00	777.55
Total	$5,295.00	$3,459.76

Gross Profit Method of Approximating Inventory

The rate of gross profit of prior periods may be used to approximate an inventory when it is impracticable or impossible to take a physical inventory. This is accomplished by utilizing the elements involved in the computation of gross profits. In a balanced table, when all but one element is known, the unknown element is found as the amount necessary to balance the table. The gross profit on sales may be computed by setting up a merchandise account as follows:

Merchandise

Inventory, Jan. 1	$100,000.00	Sales	$350,000.00
Purchases	300,000.00	Inventory, Dec. 31	105,000.00

The gross profit would be $55,000.00, the amount necessary to bring the account into balance. Now if the inventory were not known, but the gross profit could be estimated at $55,000.00, the inventory could be determined thus:

Merchandise

Inventory, Jan. 1	$100,000.00	Sales	$350,000.00
Purchases	300,000.00		
Gross profit	55,000.00		

The inventory would be $105,000, the amount necessary to bring the account into balance.

Of course the gross profit could not be definitely ascertained without an inventory, but it could be approximated by using the

average rate of gross profit on sales of former years, if no radical variations have occurred in this rate and if there is no reason to believe that the rate of the current period has been radically different from the average rate of the past.

To illustrate, let us assume that the sales and gross profits of the business whose merchandise account appears above, were as follows:

Year	Sales	Gross Profit	Per Cent
Third preceding	$200,000.00	$31,400.00	15.7
Second "	310,000.00	48,050.00	15.5
First "	385,000.00	60,170.00	15.6
	$895,000.00	$139,620.00	15.6

The annual rates are computed to determine whether there has been any considerable variation in the rates of gross profit. Then, on the assumption that the rate of gross profit for the current period was the same as the average of the rates of the three last preceding years:

15.6% of $350,000.00 (sales) = $54,600.00, approximate gross profit

Then:

Inventory, January 1		$100,000.00
Add purchases		300,000.00
Total		$400,000.00
Deduct cost of goods sold (approximate):		
Sales	$350,000.00	
Less estimated gross profit	54,600.00	295,400.00
Inventory, Dec. 31 (approximate)		$104,600.00

The inventory thus computed is $400 less than that shown by the merchandise account.

The following problem from a C. P. A. examination will further illustrate the method, the principal uses of which are in

approximating the value of merchandise destroyed by fire and in applying the gross profit test to the verification of an inventory.

Problem

The accountant is called on to confirm the inventory of a mercantile establishment. Investigation shows that inventories have been incorrectly taken and are padded. It is mutually agreed that all inventories, except the first one, which is to be used as the basis, shall be entirely ignored.

The accountant is to ascertain, on a fixed percentage of profit which it is decided shall be 33⅓% of sales, what stock should be on hand December 31, 1914, with the following data obtained from the various books:

Inventory referred to as a basis, January 1, 1911	$47,350.29
Gross purchases for the year ending December 31, 1911	76,320.15
Returned purchases	4,350.16
Freight and drayage on purchases	325.14
Gross sales	115,469.31
Returned sales	1,317.12
Gross purchases for the year ending December 31, 1912	$65,506.80
Returned purchases	3,715.16
Freight and drayage on purchases	417.15
Gross sales	105,716.10
Returned sales	1,215.84
Gross purchases for the year ending December 31, 1913	$62,517.10
Returned purchases	1,314.17
Freight and drayage on purchases	316.17
Gross sales	101,317.18
Returned sales	1,216.06
Gross purchases for the year ending December 31, 1914	$58,715.16
Returned purchases	287.50
Freight and drayage on purchases	290.10
Gross sales	95,371.16
Returned sales	417.11

SOLUTION: Since the rate of gross profit was constant throughout the four years, and since only the final inventory is required by the problem, the data can be summarized and the four years' totals used in the inventory calculation.

SUMMARY 1911–1914

YEAR	PURCHASES	RETURNED PURCHASES	FREIGHT	SALES	RETURNED SALES
1911	$76,320.15	$4,350.16	$325.14	$115,469.31	$1,317.12
1912	65,506.80	3,715.16	417.15	105,716.10	1,215.84
1913	62,517.10	1,314.17	316.17	101,317.18	1,216.06
1914	58,715.16	287.50	290.10	95,371.16	417.11
Total	$263,059.21	$9,666.99	$1,348.56	$417,873.75	$4,166.13

STATEMENT OF APPROXIMATION OF INVENTORY

At December 31, 1914

Inventory, January 1, 1911		$47,350.29	
Add cost of goods purchased, 1911–1914:			
Purchases	$263,059.21		
Less returned purchases	9,666.99	$253,392.22	
Add freight	1,348.56	254,740.78	$302,091.07
Deduct cost of goods sold, 1911–1914:			
Sales	$417,873.75		
Less returned sales	4,166.13	$413,707.62	
Less gross profit (33⅓% of sales)		137,902.54	275,805.08
Inventory, December 31, 1914 (estimated)			$26,285.99

Analysis of Statements

The following statements indicate the use which may be made of percentage in analyzing the financial statements of a business to show such facts as the ratio of cost of sales, expenses and profits to sales; the relative cost of the various elements of manufactured goods, and the variation in operating costs of different years.

Problem

From the following data obtained from the books of Johnson and Company, construct a profit and loss statement showing cost of goods manufactured, and cost and gross profit of the goods sold. Also show percentage of each element based upon cost of manufacture and based upon sales.

Raw material, January 1, 1915	$42,000,00
" " December 31, 1915	45,000.00
" " purchases during 1915	130,000.00
Freight inward	4,218.00
Wages (productive)	70,000.00
Sundry manufacturing expenses	3,500.00
Sales	280,000.00
Finished goods, January 1, 1915	18,000.00
" " December 31, 1915	22,000.00
Selling expenses	22,000.00
Administrative expenses	20,000.00

SOLUTION:

JOHNSON AND COMPANY

PROFIT AND LOSS STATEMENT

Year Ending December 31, 1915

				% OF COST	% OF SALES
Sales			$280,000.00		100.0
Deduct:					
Cost of goods sold:					
Raw material:					
Inventory, Jan. 1, 1915	$ 42,000.00				
Purchases, 1915	130,000.00				
Total	$172,000.00				
Inventory, Dec. 31, 1915	45,000.00	$127,000.00		62.0	
Freight inward		4,218.00		2.1	
Productive labor		70,000.00		34.2	
Manufacturing expense		3,500.00		1.7	
Cost of goods manufactured		$204,718.00		100.0	
Deduct:					
Inventory variation—finished goods:					
December 31, 1915	$22,000.00				
January 1, 1915	18,000.00	4,000.00			
Cost of goods sold			200,718.00		71.7
Gross profit on sales			$79,282.00		28.3
Deduct selling expenses			22,000.00		7.9
Net profit on sales			$57,282.00		20.4
Deduct administrative expenses			20,000.00		7.1
Net profit on operations			$37,282.00		13.3

This illustration, showing the per cent of net profit and gross profit on sales raises the question whether sales or cost of sales should be used as the base in the computation of rates of gross profit. In common parlance, when a statement is made that a sale has resulted in realizing a certain rate of profit, the rate is understood to have been applied to the cost. Thus, if it is said that an article costing $2 was sold at a 10% profit, one assumes that the profit was 20 cents and the selling price $2.20. But in percentage analyses of revenue statements it is much more convenient to use the net sales as the base. Selling expenses normally are proportionate to sales and the per cent of selling expense is computed on the basis of sales. By computing the cost of goods sold and all other deductions from sales as percentages of the net sales, the statement begins with 100% as the base and continues on the same basis throughout. But if cost were taken as the base, the sales would be represented by a rate exceeding 100%, and the selling expenses, including such items as advertising, salesmen's commissions, freight out and store expense, would have to be rated on the illogical basis of cost.

Therefore the percentage analysis of the revenue statement should properly be made on two bases: elements of manufacturing cost, including material, labor and factory overhead, should be considered as percentages of cost; and all deductions from sales, including the cost of goods sold, the selling expenses and the administrative expenses, should be rated as percentages of the net sales. This method was followed in the preceding illustration.

As an illustration of the use of percentage in the comparison of successive revenue statements, the following condensed statements of the J. E. Smith Wire and Iron Company are presented, together with comparative percentage analyses thereof. The analytical statement shows that the increase in sales and the increase in cost of sales have not been proportionate, as the proportion of cost to sales has steadily increased, causing a corresponding decrease in the percentage of profit.

Year	Net Sales	Cost of Sales	Selling Expense	Administrative Expense	Net Profit
1909.....	$350,000	$200,000	$50,000	$15,000	$85,000
1910.....	400,000	250,000	60,000	20,000	70,000
1911.....	450,000	300,000	70,000	25,000	55,000
1912.....	550,000	400,000	80,000	30,000	40,000
1913.....	500,000	360,000	75,000	15,000	50,000

J. E. Smith Wire and Iron Company

(Analysis of Condensed Revenue Statements, 1909 to 1913)

Year	Net Sales	Cost of Sales	% of Cost to Sales	Gross Profit on Sales	% of Gross Profit	Selling Expenses	% of Selling Expenses	Net Profit on Sales	% of Net Profit on Sales	Administrative Expense	% of Sales	Net Profit	% of Net Profit
1909.....	$350,000	$200,000	57.14	$150,000	42.86	$50,000	14.29	$100,000	28.57	$15,000	4.29	$85,000	24.28
1910.....	400,000	250,000	62.50	150,000	37.50	60,000	15.00	90,000	22.50	20,000	5.00	70,000	17.50
1911.....	450,000	300,000	66.67	150,000	33.33	70,000	15.55	80,000	17.78	25,000	5.56	55,000	12.22
1912.....	550,000	400,000	72.73	150,000	27.27	80,000	14.55	70,000	12.72	30,000	5.45	40,000	7.27
1913.....	500,000	360,000	72.00	140,000	28.00	75,000	15.00	65,000	13.00	15,000	3.00	50,000	10.00

As another illustration of the utility of percentage as a means of measuring variations, the following condensed statement of comparative manufacturing costs is given.

CONDENSED STATEMENT OF MANUFACTURING COSTS

	YEAR 1916		YEAR 1917	
	DOLLARS	% of Total	DOLLARS	% of Total
Raw material	$320,000.00	31.07	$287,600.00	37.06
Direct labor	441,500.00	42.86	301,300.00	38.83
Manufacturing expense	268,500.00	26.07	187,100.00	24.11
Total	$1,030,000.00	100.00	$776,000.00	100.00

While this statement shows the variation in the rate per cent of each item to the total cost, it does not give an adequate idea of the variation in unit costs. The following table shows this variation.

COMPARATIVE STATEMENT OF UNIT COSTS

Units Produced:
1916 10,000
1917 8,000

	YEAR 1916		YEAR 1917		INCREASE OR DECREASE*	
	Total	Unit	Total	Unit	Dollars	Per cent
Materials	$320,000.00	$32.00	$287,600.00	$35.95	$3.95	12.34%
Direct labor	441,500.00	44.15	301,300.00	37.66	6.49*	14.70*
Manufacturing expense	268,500.00	26.85	187,100.00	23.39	3.46*	12.89*
Total	$1,030,000.00	$103.00	$776,000.00	$97.00	$6.00*	5.83*

Percentage Analyses to Determine Causes of Variation in Profits

Revenue statements may be compared on a percentage basis to show the cause of the increase or decrease in net profit, but in order to arrive at accurate results it is necessary to know the per

cent of increase or decrease in the unit selling price of the second year as compared with the unit selling price of the first year. To illustrate, let us assume the following statements each of which shows a percentage analysis based on sales:

THE WHARTON MANUFACTURING COMPANY

COMPARATIVE PROFIT AND LOSS STATEMENTS

		Year Ending December 31, 1916			Year Ending December 31, 1917	
Sales		$400,000	100.00%		$600,000	100.00%
Deduct:						
Cost of sales:						
Material	$100,000		25.0	$150,000		25.0
Labor	200,000		50.0	330,000		55.0
Manufacturing expense	50,000		12.5	60,000		10.0
Total		350,000	87.50		540,000	90.0
Gross profit on sales		$50,000	12.50		$60,000	10.0
Deduct selling expense		10,000	2.50		15,000	2.5
Net profit on sales		$40,000	10.00		$45,000	7.5
Deduct general expenses		15,000	3.75		15,000	2.5
Net profit		$25,000	6.25		$30,000	5.0

This comparative statement has very little meaning until the further fact is known that there was a 20% advance in the selling price of all goods in 1917. With this additional information, the sales of 1917 can be reduced to the value which they would have brought in 1916, thus:

$600,000 ÷ 120% = $500,000, the price which the same goods would have sold for in 1916

A supposititious profit and loss statement for 1917 may now be drawn up, beginning with sales of $500,000 (volume of 1917 at 1916 prices) and replacing the actual figures of 1917 with amounts obtained by multiplying $500,000 by the various rates per cent shown in the 1916 statement. This supposititious statement will show what the figures for 1917 would have been if there had been merely a change in volume of business but no change in the rates of the various expenses to sales. A comparison of these supposititious figures with the actual figures for 1917 will show the

effect on 1917 profits of fluctuations in cost of manufacture and of expenses.

PROFIT AND LOSS STATEMENT OF 1917 REDUCED TO 1916 RATES AND COMPARED WITH ACTUAL STATEMENT FOR 1917

	Statement on Basis of 1916 Rates to Sales			Variations of Profit			
		Amounts		Decreases		Increases	
	Rates of 1916	at 1916 Rates	Actual 1917	Dollars	%	Dollars	%
Sales	100.00%	$500,000	$600,000			$100,000	20.0
Deduct:							
Cost of goods sold:							
Material	25.00	125,000	150,000	$25,000	20.0		
Labor	50.00	250,000	330,000	80,000	32.0		
Manufacturing expense	12.50	62,500	60,000			2,500	4.0
Total	87.50	$437,500	$540,000	$102,500	23.43		
Gross profit	12.50	$62,500	$60,000	$2,500	4.00		
Deduct selling expenses	2.50	12,500	15,000	2,500	20.00		
Net profit on sales	10.00	$50,000	$45,000	$5,000	10.00		
Deduct general expenses	3.75	18,750	15,000			3,750	20 0
Net profit	6.25	$31 250	$30,000	$1,250	4.00		

The additional profit of 1917 can now be accounted for as follows:

12½% gross profit (1916 rate) on $100,000.00 additional business done in 1917		$12,500.00
Deduct excess of additional cost to manufacture over increase in selling prices:		
Additional cost of material	$25,000.00	
Additional cost of labor	80,000.00	
Total	$105,000.00	
Less decrease in cost of manufacturing expense	2,500.00	
Net excess in manufacturing cost	$102,500.00	
Less increase in selling prices	100,000.00	2,500.00
Excess of gross profit of 1917 over 1916		$10,000.00
Deduct additional selling expenses of 1917		5,000.00
Additional net profit of 1917		$5,000.00

CHAPTER VII

EQUATIONS IN THE SOLUTION OF PROBLEMS

Solving Equations

Problems may often be solved by stating the conditions in the form of an equation and solving the equation by applying one or more of the following processes:

1. Multiplying both sides of the equation by the same number
2. Dividing both sides of the equation by the same number
3. Adding the same number to both sides of the equation
4. Subtracting the same number from both sides of the equation

Illustration 1

A manufacturer produced a certain commodity which he sold the first year at a certain price; he raised the price 25% the second year; increased that price 20% the third year; and in the fourth year he increased the third-year price by $16\frac{2}{3}\%$. The price the fourth year was $35. What was the price the first year?

SOLUTION: In order to obtain an equation it is necessary to represent some value by 100%. The value chosen to be represented by 100% will depend on the conditions of the problem; where convenient, 100% should represent the value required by the problem. In this case—

Let 100% = the selling price the first year
then 100% × 125% = 125% the selling price the second year
and 125% × 120% = 150% " " " " third "
and 150% × $116\frac{2}{3}\%$ = 175% " " " " fourth "

Since the selling price the fourth year was $35 we obtain the equation:

$$175\% = \$35$$

But it is desired to determine 100%, which is accomplished by dividing both sides of the equation by 1.75:

100% = $20, the selling price the first year

Illustration 2

A entered into partnership with B and was to act as manager of the business. Before dividing profits equally with B, A was to receive a special bonus of 25% of the net profit. Before calculating A's commission, the profits were shown by the revenue statement to be $5,000. How should the $5,000 be divided between A and B?

SOLUTION: This problem illustrates the difficulty which frequently arises in interpreting contracts which provide that commissions and bonuses shall be determined as percentages of the net profit. The difficulty arises from the uncertainty as to what is the amount of the net profit; for, if the bonus is to be considered as an expense of the business, the net profits are less than $5,000; if the bonus is not to be considered as an expense but as part of the distribution of profits, the net profit is $5,000. From the statement of the problem, it is impossible to tell whether or not the commission is to be considered as an expense; hence it is necessary to give two solutions.

Assuming the bonus is not an expense:

Since the bonus is 25% of the net profit, the net profit must be 100%. Since the bonus is not an expense, the $5,000 is all to be considered net profit—

Then, 100% = $5,000, the net profit
and 25% = 1,250, the bonus
and 75% = $3,750, the remaining profit to be divided equally.
37½% = $1,875, share to A and B each

Therefore, the division is as follows:

	A	B	TOTAL	RATE
Bonus	$1,250		$1,250	25%
Remainder, ½ each	1,875	$1,875	3,750	75%
Total	$3,125	$1,875	$5,000	100%

Assuming that the bonus is an expense to be deducted from the $5,000 to obtain the net profit:

	25%	= bonus
	100%	= net profit
	125%	= bonus plus profit, or $5,000
Then	100%	= $4,000, net profit
and	25%	= $1,000, bonus
Proof	125%	= $5,000

The division of the $5,000 on this assumption would be:

	A	B	Total	Rate
Gross profit			$5,000	125%
Deduct bonus	$1,000		1,000	25%
Net profit, ½ each	2,000	$2,000	$4,000	100%
Total distribution	$3,000	$2,000		

The net difference of $125 in the distribution under the two interpretations emphasizes the necessity for care in drawing up contracts of this character. In such cases an accountant should always be consulted as to the wording of the contract, as any competent accountant would recognize the danger of there being two constructions placed on the contract, with a consequent dispute.

Illustration 3

(From Ohio C. P. A. Examination, October, 1919)

The American Manufacturing Company commenced business on January 1, 1918, with a paid-up cash capital equal to the sales for the year 1918.

The net profits for the year 1918 were $26,100.

Of the total charges to manufacturing during the year, 40% was for materials, 30% for productive labor, and 30% for manufacturing ex-

penses (including 5% depreciation on plant and machinery, amounting to $3,000).

The value of the materials used was 80% of the amount purchased, and 90% of the amount purchased was paid during the year.

The inventory value of finished goods on hand at December 31, 1918, was 10% of the cost of finished units delivered to the warehouse, and the work in process at that date was equal to 50% of the cost of units delivered to the warehouse.

The selling and administrative expenses were equal to 20% of the sales; also to 40% of the cost of goods sold. Ninety per cent of these expenses were paid during the year 1918. Plant and machinery purchased during the year were paid for in cash.

All labor and manufacturing expenses (exclusive of depreciation) were paid in full up to and including December 31, 1918.

Of the total sales for the year, 80% was collected and 1% charged off as worthless.

From the given data you are required to prepare a balance sheet and a profit and loss statement, showing cost of goods delivered to the warehouse, cost of goods sold, and net profit for the year.

SOLUTION:

Let sales equal		100%
Then (since selling and administrative expense is 20% of sales or 40% of cost of goods sold) the cost of goods sold is half of the sales or		50%
and the gross profit is		50%
(That is, if a is 20% of x and is also 40% of y, then y must be half of x.)		
Selling and administrative expenses are equal to	20%	
and bad debts equal	1%	21%
Hence the net profit is		29%

Then, 29% = $26,100

and 100% = $90,000, sales for the year, and the cash capital at the beginning of the year.

Cost of goods sold = 50% of $90,000	$45,000
Gross profit	45,000
Selling and administrative expense	18,000
Bad debts	900

Since the inventory of finished goods at December 31, 1918, was 10% of the cost of finished units delivered to the warehouse, the cost of goods sold was 90% of the finished goods manufactured during the year.

Then, $\$45{,}000 \div 90\% = \$50{,}000$, cost of finished goods manufactured
And $\$50{,}000 - \$45{,}000 = \$5{,}000$, inventory of finished goods at December 31, 1918

Since the work in process at December 31, 1918, was 50% of the cost of finished goods delivered to the warehouse,

50% of \$50,000 = \$25,000, Work in process inventory

\$50,000,	Goods finished during 1918
25,000,	Work in process at Dec. 31, 1918
\$75,000,	Total manufacturing cost of 1918

40% of \$75,000.00 =	\$30,000,	Cost of materials used
30% of 75,000.00 =	22,500	" " productive labor
30% of 75,000.00 =	22,500	" " manufacturing expense

Of this manufacturing expense, \$3,000 was depreciation on plant and machinery; hence the manufacturing expense paid in cash was \$19,500. Since the rate of depreciation was 5%, the cost of plant and machinery was \$60,000, all of which was paid for in cash.

Since 80% of the material purchased was used in manufacturing, $\$30{,}000 \div 80\% = \$37{,}500$, the cost of the material purchased; and \$7,500 is the inventory of raw material at December 31, 1918. Also 90% of \$37,500, or \$33,750, is the amount of cash paid for purchases; and $\$37{,}500 - \$33{,}750 = \$3{,}750$, the accounts payable at December 31, 1918, for purchases.

The selling and administrative expenses were \$18,000. Of this amount, 90%, or \$16,200, was paid in cash. The remainder, \$1,800, is an addition to the accounts payable.

80% of the sales of \$90,000 were collected. 1% was written off. Hence,

Sales		\$90,000
Less:		
Cash collections	\$72,000	
Bad debts	900	72,900
Balance of accounts receivable		\$17,100

The cash summary is:

Cash capital paid in	$90,000	
Collections on accounts receivable	72,000	$162,000
Deduct:		
Plant and machinery	$60,000	
Materials	33,750	
Productive labor	22,500	
Manufacturing expense	19,500	
Selling and administrative expense	16,200	
Total disbursements		151,950
Balance		$ 10,050

AMERICAN MANUFACTURING COMPANY

TRIAL BALANCE

December 31, 1918

Capital stock		$90,000
Plant and machinery	$60,000	
Reserve for depreciation, plant and machinery		3,000
Sales		90,000
Purchases	37,500	
Productive labor	22,500	
Manufacturing expense	19,500	
Depreciation, plant and machinery	3,000	
Selling and administrative expense	18,000	
Bad debts	900	
Accounts receivable	17,100	
Accounts payable (3,750 + 1,800)		5,550
Cash	10,050	
	$188,550	$188,550

Inventories: raw material, $7,500; goods in process, $25,000; finished goods, $5,000.

AMERICAN MANUFACTURING COMPANY

PROFIT AND LOSS STATEMENT

Year Ending December 31, 1918

Sales			$90,000.00
Deduct:			
Cost of goods sold:			
Material:			
Purchases	$37,500.00		
Less inventory, Dec. 31, 1918	7,500.00	$30,000.00	
Productive labor		22,500.00	
Manufacturing expense		19,500.00	
Depreciation—plant and machinery		3,000.00	
Total manufacturing cost		$75,000.00	
Deduct goods in process—Dec. 31, 1918		25,000.00	
Cost of finished goods manufactured		$50,000.00	
Deduct inventory finished goods—Dec. 31, 1918		5,000.00	45,000.00
Gross profit on sales			$45,000.00
Deduct:			
Selling and administrative expense		$18,000.00	
Bad debts		900.00	18,900.00
Net profit			$26,100.00

AMERICAN MANUFACTURING COMPANY

BALANCE SHEET

December 31, 1918

ASSETS			LIABILITIES	
Plant and machinery ...	$60,000.00		Capital stock	$ 90,000.00
Less depreciation	3,000.00	$ 57,000.00	Surplus	26,100.00
Raw material		7,500.00	Accounts payable	5,550.00
Goods in process		25,000.00		
Finished goods		5,000.00		
Accounts receivable		17,100.00		
Cash		10,050.00		
		$121,650.00		$121,650.00

Illustration 4

As another illustration of the use of percentage in the solution of problems, the following C. P. A. problem is given.

The Orinoco Coal Company was incorporated under the laws of the state of Illinois, with an authorized capital of $8,000, divided into eighty shares of the par value of $100 each, which were subscribed for as follows:

Samuel Black	60 shares
William Green	10 "
John White	10 "

Samuel Black was elected president; George Brown, vice-president and manager, and Charles Pinck, secretary and treasurer. Neither Brown nor Pinck held any stock, but were to receive in addition to their salaries a percentage of the profits after charging off all losses from whatever source —Brown 15% and Pinck 10%. These shares in the profits were not to be considered an expense deductible to obtain the basis of the bonuses.

At the end of the year a meeting of the stockholders was held and the following balance sheet was presented:

ASSETS		LIABILITIES	
Cash	$ 8,031.12	Accounts payable	$ 740.22
Accounts receivable ..	817.32	Reserve for bad debts	376.05
Coal	6,644.15	Capital stock	8,000.00
Charles Pinck	2,264.14	Undivided profits	8,640.46
	$17,756.73		$17,756.73

It was announced that Treasurer Pinck, who was not financially responsible and was not bonded, had disappeared and his account was uncollectible, but would be reduced by crediting the account with his share of the net profits.

What amounts should Pinck and Brown receive as salary addition?

What is the amount of Pinck's defalcation?

If the remaining profit is divided among the stockholders, what dividend should each receive?

SOLUTION: Since the bonuses allowable to Pinck and Brown are to be calculated on the net profit after deducting all losses, the defalcation must be deducted from the undivided profits of $8,640.46 to obtain the net profit.

Let 100% = the net profit,
then $8,640.46 − defalcation = 100%.

But the amount of the defalcation is not known, except that it is the amount of Pinck's debit balance, $2,264.14, minus his 10% of the net profit; hence—

$$\$2{,}264.14 - 10\% = \text{defalcation}$$

Substituting the first term of this equation for "defalcation" in the first equation, we obtain—

$$\$8{,}640.46 - (\$2{,}264.14 - 10\%) = 100\%$$

Removing the parentheses and changing signs—

$$\$8{,}640.46 - \$2{,}264.14 + 10\% = 100\%$$

Subtracting 10% from both sides of the equation—

$$\$8{,}640.46 - \$2{,}264.14 = 90\%$$
$$\text{or } \$6{,}376.32 = 90\%$$

Dividing both sides by 90%—

$7,084.80 = 100%, the net profit
Then 708.48 = 10%, Pinck's bonus
and 1,062.72 = 15%, Brown's bonus

$2,264.14,	Pinck's debit balance	
708.48	"	bonus credited
$1,555.66	"	defalcation

	$8,640.46	undivided profit per balance sheet
	1,555.66	Pinck's defalcation
	$7,084.80	net profit—basis of Brown's and Pinck's bonus
$708.48		Pinck's bonus
1,062.72	1,771.20	Brown's bonus and total
	$5,313.60	profits remaining for dividends

$6/8$ of $5,313.60 = $3,985.20, Black's dividend
$1/8$ " " = 664.20, Green's "
$1/8$ " " = 664.20, White's "
$8/8$ " " = $5,313.60, total (as above)

CHAPTER VIII

TRADE AND CASH DISCOUNT

Trade Discount

There are two kinds of discount affecting the amount received for goods sold or the amount paid for goods bought. The first of these is trade discount, which is a device for varying prices without interfering with basic or "list" prices, often called retail prices. The convenience of its use arises from the fact that an expensive catalogue can be made permanent by recording only the list prices. The real or trade prices are determined by the trade discounts offered by the seller, which are usually contained in confidential letters or circulars sent to customers.

A great saving of expense is effected by not having to issue a new catalogue whenever market prices are modified. Moreover the printing and circulating of large catalogues would take considerable time so that it would be impossible to give effect to price revisions until weeks after the necessity for them had arisen. But a circular altering the trade discount and thus raising or lowering the actual prices, can be prepared on a reproducing machine and sent out to customers in one or two days.

The way in which a trade discount operates is as follows: A manufacturer or wholesaler sells to a retailer at 75 cents an article the list or gross price of which is $1.25. He bills the article to the buyer at the gross price but then deducts the discount of 40%, thus:

16 dozen of article	$15.00	$240.00	
Less 40%		96.00	$144.00

In this way the actual or net price of the article is established at 75 cents. This actual price is the amount entered in the books

of both seller and purchaser, neither of whom makes any record of the list price or the trade discount.

Cumulative Trade Discounts

In order to provide for fluctuations in price the device of cumulative trade discounts has been adopted. Thus the discounts quoted may be 30, 20, 10, and 5. This does not mean a total discount of 65, because each successive rate is calculated on the amount left after deducting the discount at the preceding rate. Thus, if the list price is $240, and the discount is 30, 20, 10, and 5, the computation is as follows:

List price	$240.00	
Less 30% of $240.00	72.00	$72.00
	$168.00	
Less 20% of $168.00	33.60	33.60
	$134.40	
Less 10% of $134.40	13.44	13.44
	$120.96	
Less 5% of $120.96	6.05	6.05
	$114.91	$125.09

The total discount of $125.09 is almost exactly 52 1/8% of the gross price of $240. The net or real price is $114.91.

If the wholesaler wishes to advance the price, he notifies the trade that the last discount of 5% is discontinued, which will make the amount $120.96, or that the last two discounts of 10 and 5 are replaced by 5 alone, raising the price to $127.68 ($134.40 − $6.72).

If the price is to be lowered, it is done by adding a further discount. If another 5 is added the net price becomes $109.16 ($114.91 − $5.75).

Methods of Finding Net Price

In order to avoid the necessity of making a separate computation for each discount, it is well to know how to find one rate that

will give the same result as that reached by the successive steps of the combination rate. The rule for this is:

Add the first two discounts; multiply the two discounts; subtract the second result from the first. With this result as one discount combine the third in the same manner; and take up in turn each of the other discounts.

Thus in the illustration—

The sum of .30 and .20 is	.50	
The product of .30 and .20 is	.06	.44
The sum of .44 and .10 is	.54	
The product of .44 and .10 is	.044	.496
The sum of .496 and .05 is	.546	
The product of .496 and .05 is	.0248	.5212
Therefore the combination rate is	.5212	

It is not necessary to remember any rule, because this same result can be reached by computing the discount on $100 thus:

List price	$100.00	
Less 30%	30.00	.30
	$70.00	
Less 20%	14.00	.14
	$56.00	
Less 10%	5.60	.056
	$50.40	
Less 5%	2.52	.0252
Net price and total discount rate	$47.88	.5212

This latter method has the further advantage of being susceptible of easy proof, since the net price and the discount must add to 100.

A still better way of reaching the same result is to compute the net amount at once, instead of finding the discount and deduct-

ing it. Thus, instead of ascertaining that 30% of $240 is $72 and deducting it to find $168, it is shorter to multiply 240 by 70%, or rather by .7, to get the same result, and then to multiply 168 by .8 to get 134.40, and so on. In other words, multiply each successive amount by the complement of its discount rate, and the final result will be the net price, with all discounts deducted.

In order to find a single rate for the net amount, multiply all the complements of the discount rates. Thus, the product of .7 × .8 × .9 × .95 is .4788, the single rate for the net price when the discounts are 30, 20, 10, and 5. Applying this single rate to $240 will give $114.91, the same result as by the other method.

It will save a great deal of labor, as well as insure greater accuracy, if tables are made of the net amounts resulting from the application of the discounts usually given by the concern. These tables should be made for amounts of $1 to $100 and should be built up by successive additions and not multiplications. They will prove at every tenth amount, since if the net for $1 is $.4788, it will be $4.7880 for $10, thus proving every intermediate amount, whereas there is no proof if each amount is found by independent multiplication.

It is not necessary to carry the tables beyond $100 if all the decimals are used. It is necessary only to be careful to move the decimal point enough spaces to the right to represent the higher numbers. Thus, if the list price is $4,836 and the net rate .4788, the table will show on line 36 the amount of 17.2368 and on line 48 the amount of 22.9824. Therefore:

The net of	$	36.00	is	$	17.2368
" " "		4,800.00	"		2,298.24
" " "	$	4,836.00	"	$	2,315.48

The same result could be reached with a table of only ten lines, from $1 to $10, but a table of 100 lines shortens the computations and is still easily contained on a comparatively small card.

Cash Discount

In order to induce prompt payment of accounts merchants frequently offer to deduct a certain per cent from bills if they are paid within a fixed number of days. This deduction is called a cash discount and is always applied to the net or trade price reached after all trade discounts have been deducted.

The discount terms are expressed by the figure of the per cent followed by the number of days in which the discount is allowed and then by the total number of days that may elapse before the bill becomes due, thus:

$$2/10;\ 1/30;\ N/60$$

which means that if payment is made in 10 days 2% may be deducted; if paid in 30 days 1%; and that the bill is due in 60 days net, that is, without any discount.

The advantages of giving cash discount are usually said to be that they decrease:

1. Loss from bad debts
2. Cost of collecting accounts
3. Amount of capital tied up in outstanding accounts

As only those who are comparatively strong financially are able to avail themselves of discounts offered, the first advantage would not appear to be very often realized.

The advantage to the purchaser is that he makes much more than normal interest by taking his discount. Thus, in the case of 2/10; 1/30; N/60, if the 2% is taken the purchaser gains 2% for 50 days' use of the money, which is at the rate of 14.6 per cent per annum, while if only 1% is taken he gains 1% for 30 days, or 12% per annum. If he has sufficient bank credit, he can well afford to borrow at 6 or 7% in order to take his discounts.

Discount as a Protection against Loss

The term "cash discount" is usually understood to refer to deductions that amount to a rather heavy interest. In some cases,

however, the deduction allowed is far more than this. For instance, one concern manufacturing electrical apparatus sells on terms of 40% discount if paid within 30 days. If not paid within the time the price is at list. So large a discount is not usually considered to come within the definition of a cash discount, although it is such, strictly speaking, since it is dependent upon the payment of cash. It would perhaps be better to call it a trade discount with a time limit.

A discount of this kind is adopted as a protection against loss in case of the bankruptcy of a customer. If a discount is given purely as a trade discount, a price is established that remains the same whether a bill is paid at maturity or not. If the customer becomes bankrupt, the claim filed with the receiver must be the net with all trade discounts deducted. If the list price is $1,000 and the unconditional trade discount is 40%, the claim must be filed for $600. If the final settlement is for 50%, the creditor loses $300. But if the discount has a time limit of 30 or 60 days, the time will have expired and a claim can be filed for the list price of $1,000, on which the dividend will be $500. The creditor will lose only $100, instead of $300.

Cash Discount Regarded as an Expense

A view of cash discount not very generally accepted is that the net price is the real price and that if the bill is not paid in time the discount is added as a penalty. This, of course, reverses the usual understanding of the subject, which is that discount taken is a profit. If this view is adopted, cash discount taken is eliminated entirely, and discount not taken becomes an expense.

To illustrate, if the trade price is $1,000 with an option of a cash discount of 2%, the entries would be as follows:

Purchases	$980	
Cash discount	20	
Creditor		$1,000

Then if discount is taken,

Creditor	$1,000	
Cash		$980
Cash discount		20

In this case the discount disappears entirely.

If the discount is not taken the entry would be:

Creditor	$1,000	
Cash		$1,000

This leaves the charge of $20 in the cash discount account as an expense.

CHAPTER IX

TURNOVER

Indefinite Meaning of "Turnover"

The principal difficulty in discussing turnover is to obtain a clear idea of what is meant by the term.

In spite of the efforts of several committees of the American Association of Public Accountants, we are no nearer an authoritative standard of accounting terminology than we were ten years ago. The principal difficulty in arriving at correct definitions is that few authors on accounting subjects attempt any definitions at all. They seem to take for granted that everyone else attaches the same meaning to a term as they do themselves and that a definition of it is, therefore, unnecessary. There does not seem to be any formal definition of "turnover" in any standard work on accounting, and the word does not appear to be used with any clearcut meaning.

R. H. Montgomery comes nearest to defining the word, but even he only suggests a definition. After stating that authorities differ greatly as to what the term means, he says: "Uniformity is desirable in accounting terminology, so the author suggests this definition: The turnover of a merchant or manufacturer represents the number of times his capital in the form of stock-in-trade is re-invested in stock-in-trade during a given period."[1]

This is the generally accepted definition of turnover; that is, the number of times the merchandise is turned over. Mr. Montgomery further says: "To ascertain the turnover, take the starting inventory, add the purchases or cost of manufactured goods, and deduct the inventory at the end; divide the total by the start-

[1] R. H. Montgomery, Auditing Theory and Practice, 1919, p. 455.

ing inventory. The result will be the number of times the capital invested in stock-in-trade has been turned over during the period. The calculations are based upon a normal inventory."[2]

Normal Inventories Necessary

The application of this rule to two successive years of a business will exhibit the importance of the qualification that the inventories must be normal. It will also show the difficulty of determining in the case of the majority of concerns how many times the stock-in-trade is turned over.

If the starting inventory of the first year is $25,000, the purchases $200,000, and the inventory at the end is $50,000, the turnover during the year is 7, since $25,000 + $200,000 − $50,000, or $175,000, is seven times the first inventory. If in the following year the starting inventory is $50,000, the purchases $175,000, and the ending inventory $25,000, the turnover is 4, since $50,000 + $175,000 − $25,000, or $200,000, is four times the first inventory. Thus by the mere accident of a difference in the amount of the inventories at the beginning of the two years, the second year, which did the larger business, shows only a little more than half of the turnover of the first year. As the number of turnovers in the year is supposed to be a measure of the prosperity of the business, this method of determining it is evidently unsatisfactory.

The difficulty is greatly reduced if it is possible to determine the average, normal quantity of stock-in-trade carried. This normal quantity may or may not be the same as the inventory at the beginning of the year as there is no necessary connection between the two. Usually, however, we only know that there was a total turnover of $175,000 or $200,000 during the year. Not knowing the amount of the concern's normal inventory from any figures contained in either its revenue statement or balance sheet, we have no means of finding how many times the stock has been

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[2] R. H. Montgomery "Auditing Theory and Practice," 1919, p. 455.

turned over. This we can ascertain only if inventories are taken monthly or at other intervals throughout the year, the average of which may be regarded as the normal inventory. In any event it is necessary to define the word "normal" as it may mean the average inventory that should be carried or the inventory that is customarily carried.

Different Bases of Comparison

Even if this difficulty is met, another presents itself when choosing a basis for comparing the turnovers of two concerns. Mr. Montgomery probably represents at least the majority of American accountants in considering turnover to be the number of times the normal stock-in-trade is reinvested in the goods sold. British accountants, on the other hand, following Lisle, say that it should be the relation between the inventory and the sales. It can readily be seen that a serious misunderstanding results in the comparison of the turnovers of two concerns if one is calculated on the basis of cost and the other on the basis of sales, especially if high selling expenses make the cost of the goods a comparatively small part of the selling price.

This difference of opinion as to the proper basis for comparing turnovers has reference to trading businesses. In the case of manufacturing businesses the disagreement is even greater, as three bases of comparison may be used, raw material, raw material plus labor, and the total cost of the goods sold.

Working Capital as Basis of Turnover

Owing to this wide divergence of opinion, it has been suggested, and it would seem justifiably, that a better basis for calculating the turnover is the working capital of the business. If this method is adopted, most of the difficulties encountered in computing the turnover disappear, because the amount of working capital is indicated in the balance sheet, being the excess of current assets over current liabilities. Except as affected by the

slight increase due to undistributed profits, it remains the same throughout the entire period, and, therefore, affords a stable basis of comparison.

Every business, whether it is engaged in trading or manufacturing, requires capital for two purposes: first, to provide the necessary fixed assets to carry on the operations; and second, to furnish sufficient funds to carry the stock-in-trade and accounts receivable until cash is realized on them and more goods or material for manufacture are bought. The point that interests the proprietor is how many times a year he can invest his available floating capital by repeating the process of buying the goods, selling them, and collecting the proceeds. If one person with a working capital of $40,000 is able to sell in a year goods costing $200,000, while another with the same amount of working capital is able to sell goods costing only $160,000, the first person has turned over his capital five times to the other person's four times.

Some exception is taken to this definition of turnover. It is said that working capital is used for many purposes, including advertising campaigns and similar items of expense that do not affect the cost of manufacture. It is also objected that accounts receivable vary to a great extent between different concerns, as some sell on short time and others on long time, and some keep a big stock of raw materials, while others do not consider it necessary, or cannot afford to do so.

The question of what constitutes the proper basis for reckoning the turnover seems to turn on the object for which the turnover is used. If it is merely for the purpose of showing that one manager can handle more goods than another with the same average amount of stock on hand, the proper test is the relation between the normal inventory and the cost of the goods sold. But a manager could establish a record on this basis by following the foolish policy of buying in small quantities at retail prices and paying the high expressage instead of the low freight rates. His turnover would be large, but would lead to disaster.

If a profitable business is the object sought for, the use the manager makes of the working capital would seem to be one of the best measures of his success. If too much of the working capital is diverted from the production or purchase of goods for sale to carrying on an extensive advertising campaign, or if it is temporarily locked up in long-time accounts receivable, the quantity of goods sold is apt to decrease. This will show itself in a lessened turnover of working capital. On the other hand, if either the beginning or the normal inventory is made the measure, the turnover will remain the same even though the business done is smaller, because the inventory will decline with the volume of business. An inventory of $25,000 and cost of sales totaling $100,000 will show the same turnover on this basis as $50,000 inventory and cost of sales totaling $200,000. When the yardstick varies in length, comparative measurements are of little value.

The use of working capital as the basis of turnover is logical, first, because the capital is put in the business for the purpose of being turned over as rapidly as possible; second, because it is virtually constant; and third, because it presents all the elements concerned in the turnover, not only the stock-in-trade, but also the accounts and notes receivable, by means of which the turnover is effected. The turnover of working capital also furnishes a better criterion of the excellence of the management. With the inventory as the only standard a manager can make an apparently good record by starving his stock-in-trade. If, however, he uses working capital as the standard, he makes his best record by diligence in collecting outstanding accounts, and increasing the supply of cash for the development and handling of a more extensive business.

Definition of Working Capital

Even if this is granted, the difficulty does not come to an end, because there is no authoritative definition of working capital.

H. R. Hatfield says: "Working capital has long had a specific

meaning as a collective term for what are often called quick assets, e. g., cash, accounts receivable, perhaps merchandise, etc."[3]

H. C. Bentley says: "Working capital is the excess of quick assets over quick liabilities."[4]

Each of these authorities has his followers, as was shown in a recent discussion of the subject in the *Journal of Accountancy*. The definitions differ because in one the notes and accounts payable are considered to be borrowed capital, while in the other only the amount contributed by the proprietor is treated as capital. The first is the economic view of what constitutes capital, while the second is the business and general accounting view. If the ordinary business man is asked how much capital he has in his business, he will always state the amount of his proprietary interest. Unlike Micawber, he does not think he has added to his capital whenever he issues a note payable.

Need of Exact Definitions

The final lesson to be learned from the consideration of this subject is that in preparing comparative statements of turnover, accountants should first define the terms used and should plainly state the basis of the calculations. Otherwise the conclusions reached will be entirely misleading to persons whose conception of the subject is different. It is to be hoped that the Institute of Accountants will eventually end the present ambiguity by formulating an authoritative definition of turnover.

Having settled upon the basis to be used, the accountant determines the turnover by ascertaining how many times the normal inventory will go into the cost of the goods sold, or into the sales, according to which view is adopted, or how many times the working capital has been reinvested in the purchase or production of goods sold. The greater the quotient in each case, the more prosperous and better managed the business is supposed to be.

[3] H. R. Hatfield, Modern Accounting, 1909, p. 179.
[4] H. C. Bentley, Science of Accounts, 1911.

CHAPTER X

PARTNERSHIPS

Division of Profits

Profits may be divided by partners in any proportions to which they agree; if they make no express agreement the law implies an agreement to divide the profits equally, regardless of the capital or services contributed. The customary methods of dividing profits are:

1. In the ratio of the capital balances at the beginning of the period.
2. In the ratio of the average capitals for the period.
3. In an arbitrary ratio, usually expressed in terms of fractions or per cents.
4. In an arbitrary ratio after allowing interest on the capitals.

To illustrate these methods, assume the following facts:

Illustration

A and B are in partnership and the profits for division at the end of the year are $12,000.

A's capital account during the year undergoes the following changes:

Credit balance, January 1	$50,000	
Investment, March 1	2,000	
" November 1	3,000	
Total credits		$55,000
Withdrawal, May 1	$500	
" December 1	1,000	
Total debits		1,500
Credit balance, December 31		$53,500

B's capital account changes as follows:

Credit balance, January 1	$25,000	
Investment, February 1	10,000	
" June 1	5,000	
Total credits		$40,000
Withdrawal, October 1		1,000
Credit balance, December 31		$39,000

SOLUTION 1: *Division of profits in the ratio of the capital balances at the beginning of the year:*

This ratio is A, 50, and B, 25; or 2 to 1. A, therefore, is credited with ⅔ of $12,000, or $8,000; and B with ⅓ of $12,000, or $4,000.

SOLUTION 2: *Division of profits in the ratio of the average capital for the period:*

This ratio may be computed in either of two ways. The first is as follows: Multiply each capital account credit by the number of months or days from the date of the credit until the end of the period, and find the sum of these products. Multiply each capital account debit by the number of months or days from the date of the debit until the end of the period, and find the sum of these products. Find the difference between the credit products and the debit products.

Do this with each capital account and determine what fraction each difference is of the sum of the differences. The difference between the credit and debit products of A's capital account is found thus:

CREDITS

DATE	AMOUNT	TIME	PRODUCT	
January 1	$50,000	12 mo.	$600,000	
March 1	2,000	10 "	20,000	
November 1	3,000	2 "	6,000	$626,000

DEBITS

May 1	$ 500	8 mo.	$ 4,000	
December 1	1,000	1 "	1,000	5,000
Difference				$621,000

The difference between the credit and debit products of B's capital account is found thus:

CREDITS

DATE	AMOUNT	TIME	PRODUCT	
January 1.....	$25,000	12 mo.	$300,000	
February 1.....	10,000	11 "	110,000	
June 1.....	5,000	7 "	35,000	$445,000

DEBITS

October 1.....	$1,000	3 mo.		3,000
Difference.....				$442,000

The partnership profits of $12,000 are accordingly divided in the following ratios and amounts:

RATIOS EXPRESSED IN FRACTIONS	DIVISION OF PROFITS
A.....621/1063	$ 7,010.35
B.....442/1063	4,989.65
	$12,000.00

The second method by which profits may be divided in the ratio of the average capitals for the period is as follows:

Multiply the opening balance of each account by the number of months or days it remained unchanged.

Multiply each new balance resulting from investments or withdrawals by the number of months or days it remained unchanged.

Find the sum of these products for each capital account.

Find the ratio of each sum to the total for all.

The products for A's capital account are computed in the following way:

Balance	From		To		Time	Product
$50,000	January	1	March	1	2 mos.	$100,000
52,000	March	1	May	1	2 "	104,000
51,500	May	1	November	1	6 "	309,000
54,500	November	1	December	1	1 mo.	54,500
53,500	December	1	December	31	1 "	53,500
					12 mos.	$621,000

The products for B's capital account are figured as follows:

Balance	From		To		Time	Product
$25,000	January	1	February	1	1 mo.	$ 25,000
35,000	February	1	June	1	4 mos.	140,000
40,000	June	1	October	1	4 "	160,000
39,000	October	1	December	31	3 "	117,000
					12 "	$442,000

This method results in the same ratios as the first method, and it has the advantage that the final balances shown in the computation ($53,500 and $39,000) are the same as the balances of the accounts; and that the time numbers used as multipliers add to a full year. Checks on the accuracy of the computations are thus provided.

It must be understood that these computations determine the average capital ratios but not the average capitals. To compute the average capitals it would be necessary to divide by 12, thus:

$621,000 ÷ 12 = $51,750.00, A's average capital
$442,000 ÷ 12 = $36,833.33, B's " "

Since only the ratio between the average capitals is required, the division by 12, or by 365 if the numbers of days have been used as multipliers, is unnecessary.

Solution 3: *Division of profits in an arbitrary ratio.* No limit can be placed on the variety of arbitrary ratios which can be agreed upon and illustrations are unnecessary.

Solution 4: *Division of profits in an arbitrary ratio after allowing interest on capital.* No interest can be allowed unless there is a specific agreement to do so, and in that event the rate should be agreed upon.

Assuming a rate of 6% on the opening balances of the capital accounts, the distribution, if the remaining profits are divided equally, would be as follows:

	A	B	Total
6% of $50,000	$3,000		
6% of 25,000		$1,500	
Total interest			$4,500
Balance equally	3,750	3,750	7,500
	$6,750	$5,250	$12,000

If the agreement provides for interest on partners' capitals, it must be credited to them even though it exceeds the total profits. In that event the resulting debit balance in the profit and loss account is charged to the partners in the agreed ratio.

Assuming that the profits were only $4,000, the division would be as follows:

	A	B	Total
Credits for interest (as above)	$3,000	$1,500	$4,500
Debits for excess of interest over profits	250	250	500
Net credits	$2,750	$1,250	$4,000

If interest is provided for in the agreement, it must be credited to the partners even though a loss has been incurred instead of a profit earned. Assuming a loss of $1,000, the division would be as follows:

	A	B	Total
Debits for sum of loss of $1,000 and interest of $4,500	$2,750	$2,750	$5,500
Credits for interest	3,000	1,500	4,500
Net credit	$250		
Net debit		$1,250	$1,000

The result is that B bears all of the loss from operations as well as the $250 net credit to A.

Liquidation of Partnerships

When a partnership is terminated, the procedure to be followed in realizing the assets, liquidating the liabilities and distributing the partners' capitals, depends on whether all losses on realization have been ascertained before payments are made to the partners. If they have been, the losses are deducted from the partners' capitals in the profit and loss ratio, and the remaining assets, after paying the outside creditors, are distributed to the partners in amounts sufficient to pay off the capitals.

To illustrate, assume that all partnership debts have been paid, and that the capital accounts are:

A $10,000 B $8,000

There must be assets of $18,000. These are sold for $15,000. With losses divided equally, the division of cash proceeds is as follows:

	A	B	Total
Capitals	$10,000	$8,000	$18,000
Losses on realization	1,500	1,500	3,000
Balances paid in cash	$8,500	$6,500	$15,000

Periodical Distributions

If periodical distributions to the partners are made before all assets are realized and all losses ascertained, they should be made, if possible, in such a way as to reduce the balances of the capital accounts to the profit and loss ratio existing between the partners, so that if all remaining assets are lost each partner's capital account will be exactly sufficient to cover his share of the loss.

To illustrate, assume that all liabilities are paid and the partners' capitals are as follows:

A $15,000
B 20,000
C 25,000

The assets total $60,000. In realizing on $18,000 worth of assets a loss of $3,000 is incurred, so that there is $15,000 in cash to divide. The division of cash should be made as follows, assuming that the partners share profits and losses equally.

	A	B	C	Total
Capitals	$15,000	$20,000	$25,000	$60,000
Loss	1,000	1,000	1,000	3,000
Balance before dividing cash	$14,000	$19,000	$24,000	$57,000
Cash		5,000	10,000	15,000
Balances left in P. & L. ratio	$14,000	$14,000	$14,000	$42,000

It should be noted that the cash is not distributed in the capital ratio, the profit and loss ratio, nor any other ratio, but in arbitrary amounts sufficient to reduce the capitals to the profit and loss ratio.

Reducing Capitals to Profit and Loss Ratio

It is not always possible to bring the capital account balances to the profit and loss ratio at the first distribution of cash. This is the case if the capital of any partner after charging off all ascertained losses is less than his profit and loss ratio of the assets which will remain after making the proposed distribution.

To illustrate, assume the capitals to be as follows:

A	$10,000
B	20,000
C	30,000

Losses are to be shared equally. The assets total $60,000 and all liabilities are paid. Assets carried on the books at $30,000 are sold for $24,000, the loss of $6,000 being divided as follows:

	A	B	C	Total
Capitals before dividing loss	$10,000	$20,000	$30,000	$60,000
Loss	2,000	2,000	2,000	6,000
Capitals before distribution of cash	$8,000	$18,000	$28,000	$54,000

After the $24,000 is divided there will remain a total capital of $30,000. If possible the $24,000 cash should be divided in such a way as to leave each partner with a balance of $10,000; but this is clearly impossible since A's balance is already reduced to $8,000. Since A's capital is not sufficient to bear his share of the total possible loss of the $30,000 of assets which remain after the distribution, nothing should be paid to A. In the event that a total loss of $30,000 is incurred the charge of $10,000 to A would leave his account with a debit balance of $2,000. If he could not pay in the $2,000 it would have to be charged against B and C in their profit and loss ratios, which in this case happen to be equal. Therefore, B and C may possibly lose the following amounts:

	B	C
One-third each of total possible $30,000 loss	$10,000	$10,000
Excess of A's share of possible loss over his capital	1,000	1,000
Total possible loss	$11,000	$11,000

The $24,000 cash should be divided between the two partners in such a way as to reduce their balances to these amounts, as follows:

	A	B	C	Total
Capitals before distribution of cash	$8,000	$18,000	$28,000	$54,000
Cash distributed		7,000	17,000	24,000
Balances	$8,000	$11,000	$11,000	$30,000

In the next realization, assets carried at $15,000 are realized at $12,000, the loss being $3,000. After dividing the loss the cash can be distributed in amounts which will reduce the capital balances to the profit and loss ratio of equality.

	A	B	C	TOTAL
Balances (as above)	$8,000	$11,000	$11,000	$30,000
Loss	1,000	1,000	1,000	3,000
Balances before distributing cash	$7,000	$10,000	$10,000	$27,000
Cash	2,000	5,000	5,000	12,000
Balances (reduced to P. & L. ratio)	$5,000	$ 5,000	$5,000	$15,000
The remaining assets are sold for $9,000, the division of loss and cash being as follows:				
Loss	2,000	2,000	2,000	6,000
Balances paid in cash	$3,000	$ 3,000	$ 3,000	$ 9,000

CHAPTER XI

THE CLEARING HOUSE

Principle of the Clearing House

The general principle on which a clearing house operates is that of offsetting debits and credits and dealing only with net differences. Its simplest form is illustrated when two persons buy from and sell to each other. If instead of each paying his bill to the other in full, the one who owes the larger sum deducts the other's debt to him and pays the difference, he has to that extent adopted the clearing house principle.

In the fullest application of the principle, an outside party is introduced to act as settling or clearing agent. Each unit in the combination that forms the clearing house has dealings with every other unit, but instead of settling its dealings with each of the other units individually it charges the total of all its debits and credits the total of all its credits to the clearing agent, with which it settles the net difference of the totals. In this way a single settlement takes the place of a large number of settlements. An example will make this clear.

Debits and Credits with Clearing House

Suppose there are six banks in a city. Every day they receive on deposit and otherwise, checks on each other. If there is no clearing house, each bank presents to each of the others the checks drawn on it and collects or pays the difference due to or from it, according as the amount of the checks it presents is greater or smaller than the amount of those presented to it.

If the banks form a clearing house, each provides itself with a blank form, in which it inserts the amount of checks it holds

against each of the others. The total it charges to clearing house checks on the teller's blotter. This form and the checks themselves are sent to the clearing house. The checks are delivered to the representatives of the banks on which they are drawn, with a memorandum of the amount for each bank. As each bank receives the checks drawn on it, it enters the amounts in the second column of the form and the total becomes the credit to clearing house checks. If a bank's credits to the clearing house are greater than its debits, it pays the difference to the clearing house. The amounts thus paid in are afterwards paid to those banks whose debits to the clearing house are greater than their credits.

Clearing House Transactions

The following are assumed to be the transactions on a certain day of the six banks in the clearing house, it being remembered that the first column of figures on each blank is made up before the checks leave the bank and the second column at the clearing house.

FIRST NATIONAL BANK

Atlas Nat.....	$27,819.32	$16,248.76
Merchants St.....	12,948.24	14,629.83
Traders St.....	22,687.19	26,963.42
State Tr. Co.....	18,729.63	21,246.38
Union Nat.....	14,963.48	15,782.16
	$97,147.86	$94,870.55
C. H. owes.....		$ 2,277.31

ATLAS NATIONAL BANK

First Nat.....	$16,248.76	$27,819.32
Merchants St.....	11,283.42	8,664.19
Traders St.....	12,732.63	10,497.26
State Tr. Co.....	15,376.28	11,958.47
Union Nat.....	9,659.87	7,326.58
	$65,300.96	$66,265.82
Owes C. H.....	$ 964.86	

MERCHANTS STATE BANK

First Nat.....	$14,629.83	$12,948.24
Atlas Nat.....	8,664.19	11,283.42
Traders St.....	9,586.28	10,392.16
State Tr. Co.....	11,473.47	9,627.59
Union Nat.....	10,827.42	11,432.27
	$55,181.19	$55,683.68
Owes C. H.....	$ 502.49	

TRADERS STATE BANK

First Nat.....	$26,963.42	$22,687.19
Atlas Nat.....	10,497.26	12,732.63
Merchants St.....	10,392.16	9,586.28
State Tr. Co.....	11,788.39	13,249.36
Union Nat.....	9,673.57	8,865.71
	$69,314.80	$67,121.17
C. H. owes.....		$ 2,193.63

State Trust Company			Union National Bank		
First Nat.....	$21,246.38	$18,729.63	First Nat.....	$15,782.16	$14,963.48
Atlas Nat.....	11,958.47	15,376.28	Atlas Nat.....	7,326.58	9,659.87
Merchants St.....	9,627.59	11,473.47	Merchants St.....	11,432.27	10,827.42
Traders St.....	13,249.36	11,788.39	Traders St.....	8,865.71	9,673.57
Union Nat.....	7,982.25	8,842.74	State Tr. Co.....	8,842.74	7,982.25
	$64,064.05	$66,210.51		$52,249.46	$53,106.59
Owes C. H.....	$ 2,146.46		Owes C. H.....	$ 857.13	

Manager's Sheet

The representatives of the different banks are allowed a certain number of minutes in which to finish the preparation of their respective sheets. They then in turn call out their figures which are registered by the manager of the clearing house on his own sheet in the following manner:

Banks	Debits	Credits	Debit Balances	Credit Balances
First National Bank	$ 94,870.55	$ 97,147.86	$	$2,277.31
Atlas National Bank	66,265.82	65,300.96	964.86	
Merchants State Bank	55,683.68	55,181.19	502.49	
Traders State Bank	67,121.17	69,314.80		2,193.63
State Trust Company	66,210.51	64,064.05	2,146.46	
Union National Bank	53,106.59	52,249.46	857.13	
	$403,258.32	$403,258.32	$4,470.94	$4,470.94

If the manager's sheet balances, he rings his bell to signify that the session is over. If it does not balance he announces the fact. A certain number of minutes is allowed for finding the error. If found within that time the one who made the error is fined a certain sum. The fine increases progressively with any added time taken to reach a balance.

Later, the banks with debit balances make their payments to the clearing house manager, who shortly after pays off the banks with credit balances.

The total figures are kept by the manager for statistical purposes. The sum of the checks presented ($403,258.32) is the clearing house movement for the day, and the sum of the balances ($4,470.94) is the amount of the balances reported in the financial

news. The corresponding figures for the week, month and year are published periodically in the financial columns of the newspapers, and are considered a barometer of business activity.

Economy of System

The convenience and the economy of the clearing house system may be demonstrated by an analysis of the transactions of the Merchants State Bank. This bank has collected checks aggregating $55,181.19 and has paid checks totaling $55,683.68, or a total settlement of $110,864.87, by a payment of only $502.49, or less than one half of 1%. Without the clearing house, it would have been compelled, even by offsetting with each of the other banks, to make the following payments:

Atlas National Bank	$2,619.23	
Traders State Bank	805.88	
Union National Bank	604.85	$4,029.96
It would have received the following amounts:		
First National Bank	$1,681.59	
State Trust Company	1,845.88	3,527.47
Total cash movement in the bank would have been...		$7,557.43

One payment of $502.49 takes the place of five transactions aggregating fifteen times as much.

Each bank must settle on the basis of the clearing house returns. The corrections of any errors in addition, the listing of checks, either for wrong amounts or on the wrong bank, etc., and the settlement for checks returned for lack of funds or on account of missing endorsements, are all matters left to the individual banks to adjust with each other.

Application of Principle Extended

The clearing house principle is applied to the settlement of accounts whenever several parties have reciprocal relations, in-

volving the transfer of value in the form of money, securities, or merchandise. It is applied to the settlement of accounts between brokers on boards of trade and other exchanges, and also to the regulation of sales of merchandise when an agreement exists among concerns in the same line of business restricting the output of each to an agreed percentage. At present there is very little of the latter use of the method, owing to the danger of violating the law which prohibits agreements in restraint of trade.

CHAPTER XII

BUILDING AND LOAN ASSOCIATIONS

General Characteristics of Building and Loan Associations

A building and loan association is founded on the general principle of co-operation among a large number of small investors and a smaller number of intending borrowers, who wish to get advances for the purpose of erecting buildings on land they own, or of paying off existing loans on real estate that has been partially paid for. There are several different plans on which such associations are organized, but the idea of co-operation is present in them all.

In some foreign countries, notably Italy, this principle of co-operation has been extended to cover loans to workmen for the purchase of tools or machinery and even to the furnishing of large sums to working contractors for building railroads and other public works.

The usual characteristics of a building and loan association are as follows: Each member subscribes to a definite number of shares, on which he pays monthly instalments of one half of 1% of the par of the stock. If, for example, the par is fixed at $100, the monthly payment on each share is 50 cents. When these instalments together with the profits earned by each share amount to $100, the stock is said to have matured and is paid off.

Terminating Plan

The terminating plan of organization, the first to be adopted, is the simplest of all. It involves the subscription by a limited number of persons to a certain number of shares which are to be paid in instalments, and the dissolution of the association upon the

maturity of the shares. As money is paid by subscribers it is loaned from time to time. As the interest collected is also loaned, the association is in theory receiving compound interest. For instance, if 2,000 shares are taken and 50 cents per share is paid monthly, a loan of $1,000, say at 6%, can be made at the end of the first month. At the close of the second month there is another $1,000 to lend, together with $5 interest collected on the first loan, and this is repeated from month to month. The compound interest theory, however, does not hold strictly here, as it is difficult to lend odd amounts, unless other security than real estate is taken.

The objection to this plan is that it is impossible to employ the money profitably in real estate loans, because during the latter part of the association's life loans have to be made for so short a time that great difficulty is experienced in making them; and if the later loans should be made for the usual term of five years the dues and profits on the stock could not be paid when they matured. It can, however, be arranged that after the stock reaches par no further payments should be made on the shares and no shares should be withdrawn, but the money should be kept invested and cash dividends paid out of the interest collected.

Practicability of Plan

The terminating plan can be used to best advantage when a large number of persons wish to pay in a given time a large sum of money with interest, each person paying a small amount monthly for the given time. This plan, for example, is extremely useful in paying off a church debt. If a church without any rich members owes $10,000 on which it is paying 6% interest, it might seem impossible to raise so large an amount. If a building and loan association is formed with 100 shares of $100 each, to run for five years, on which the monthly payments are fixed at $1.93 per month, it will usually not be found difficult to persuade the con

gregation to take the whole 100 shares. This will not only pay off the entire debt in five years, but will also pay the interest on the diminishing amount of the note, as the holder of the note will be willing to accept partial payments of even hundreds of dollars.

The plan works out in this way—Each holder of a share agrees to pay $100 and interest in monthly instalments of $1.93. If he makes his payments regularly, his account for the first two months runs as follows:

Original agreement	$100.00
One month's interest at 6%	.50
	$100.50
First month's payment	1.93
	$ 98.57
One month's interest at 6%	.49
	$ 99.06
Second month's payment	1.93
	$ 97.13

The treasurer credits the interest account each month with the interest paid that month, and the principal with the balance of the payments, applying the money to the payment of the interest and principal of the note.

The amount of the monthly payments theoretically necessary is found from the compound interest table headed, "Amount per annum necessary to pay a debt of $1 now due and the interest thereon." In order to reduce it to monthly payments it is necessary to use the table of ½ % for the number of months in the life of the association. In our illustration we find the amount necessary, or $1.93, in the ½ % column on the line of the 60th period. For six years or 72 months the amount is $1.65¾. As all the amounts will not be paid promptly, it is better to make the monthly payments $2 and $1.70 respectively.

Serial Plan

The serial plan is virtually a succession of terminating plans combined in one association. A series is started at regular intervals of three, six or twelve months, with whatever number of shares may be subscribed. In some states the par of a share is $200 and the monthly payment per share is $1; but in most of the states the par is $100 and the payment is 50 cents per month, sometimes 25 cents per week. After a new series is begun no more shares can be issued in any previous series, although the holder of stock in any series can sell it or otherwise transfer it.

A serial association is a partnership with limited liability, the subscribers to the different series being the partners. The profits are divided among the series in proportion to their capitals which are the amounts paid in plus the accumulated profits to date. A simple illustration will make the process clear.

Suppose that an association is started on January 1, 1917, and that on September 30, 1918, the shares outstanding are as follows:

Series	Date of Issue	Number of Shares	Paid per Share	Profit per Share	Value per Share
1	January, 1917	800	$10.50	$2.29	$12.79
2	April, "	600	9.00	1.71	10.71
3	July, "	700	7.50	1.25	8.75
4	October, "	500	6.00	.88	6.88
5	January, 1918	400	4.50	.56	5.06
6	April, "	600	3.00	.27	3.27
7	July, "	500	1.50	.02	1.52

During the quarter beginning October 1, 1918, the eighth series of 700 shares is sold. The net profits for the quarter from interest, fines, etc., less the secretary's salary and other expenses, are $489.92. The undivided profits on September 30, 1918, are

$4,571.28. There are two plans by which the profits may be divided, the partnership plan and the Dexter rule.

Distribution by Partnership Plan

In the so-called partnership plan all previous profits are ignored and the capital of each series is taken as the money actually paid in multiplied by the equated number of months during which it has been left in the business. As the dues are payable on the first of each month, the equated time is found by adding 1 to the number of months the series has run and dividing the sum by 2. Thus, on December 31, 1918, the first series has run 24 months. Adding 1 and dividing by 2 gives 12½ months as the equated time. As there are 800 shares in that series and as each share has paid in $12 the capital is $12 multiplied by 800 or $9,600 and it has been in an equated time of 12½ months. Multiplying $9,600 by 12½ gives $120,000 as the equated capital of Series 1. Pursuing the same method the equated capitals of the several series are calculated as follows:

Series								
1.....	$12.00	×	12½	mos.	×	800	shares	$120,000
2.....	10.50	×	11	"	×	600	"	69,300
3.....	9.00	×	9½	"	×	700	"	59,850
4.....	7.50	×	8	"	×	500	"	30,000
5.....	6.00	×	6½	"	×	400	"	15,600
6.....	4.50	×	5	"	×	600	"	13,500
7.....	3.00	×	3½	"	×	500	"	5,250
8.....	1.50	×	2	"	×	700	"	2,100
Total of equated capitals								$315,600

The total profits to date, amounting to $5,061.20, are divided among the series in the proportion that each one's equated capital bears to the total equated capital. Reducing to the least common denominator of 2,104, we have the following division of the profits among the several series.

Series				
1....	800/2104	of $5,061.20	equals $1,924.40	$2.41 per share
2....	462/ "	" "	1,111.34	1.85 " "
3....	399/ "	" "	959.79	1.37 " "
4....	200/ "	" "	481.10	.96 " "
5....	104/ "	" "	250.17	.62 " "
6....	90/ "	" "	216.50	.36 " "
7....	35/ "	" "	84.19	.17 " "
8....	14/ "	" "	33.68	.05 " "
	2,104		$5,061.17	

Of course it is impossible to balance exactly the total profits with the sum of the profits assigned to each series.

Distribution by Dexter's Rule

The first step in applying the Dexter rule is to calculate the capitals of the series. The capital of each series on September 30 is the value per share as shown in the table on page 106 [of this chapter] multiplied by the number of shares outstanding. To find the earning capital on December 31, it is necessary to add the contribution of 50 cents per share for each of the three months of the current quarter. This is an average of $1 per share, calculated thus:

50 cents paid	October	1	was in 3 months	or	$1.50	for 1 month
50 " "	November	1	" " 2 "	"	1.00	" 1 "
50 " "	December	1	" " 1 month	"	.50	" 1 "
Total					$3.00	" 1 "

This is an average of $1 for three months. One dollar must, therefore, be added to the value per share shown in the first table. Bearing this in mind, the earning capital of the different series is found as follows:

Series							
1.....	800	shares	at	$13.79	per	share	$11,032
2.....	600	"	"	11.71	"	"	7,026
3.....	700	"	"	9.75	"	"	6,825
4.....	500	"	"	7.88	"	"	3,940
5.....	400	"	"	6.06	"	"	2,424
6.....	600	"	"	4.27	"	"	2,562
7.....	500	"	"	2.52	"	"	1,260
8.....	700	"	"	1.00	"	"	700
Total capital.....							$35,769

This rule leaves undisturbed the previous distributions of profits and adds to them the profits of the current quarter on the basis of the present earning capitals. The current profits of $489.92 are 1.37 per cent of the capital of $35,769. Hence the profit per share for each series is calculated as follows:

Series	Capital	Current Profit	Previous Profit	Total Profit	Shares	Profit per Share
1.....	$11,032	$151.14	$1,832	$1,983.14	800	$2.48
2.....	7,026	96.26	1,026	1,122.26	600	1.87
3.....	6,825	93.50	875	968.50	700	1.38
4.....	3,940	53.98	440	493.98	500	.99
5.....	2,424	33.21	224	257.21	400	.64
6.....	2,562	35.10	162	197.10	600	.33
7.....	1,260	17.26	10	27.26	500	.05
8.....	700	9.59		9.59	700	.01
	$35,769	$490.04	$4,569	$5,059.04		

A comparison of the profits per share shows that the partnership plan gives larger profits to the later at the expense of the earlier series. This is a great injustice, as the earlier series are made to share the profits they have accumulated, with the later series, which had no part in producing those profits. This is not done in an ordinary commercial business when new partners are admitted. Dexter's rule is the true partnership plan and is the one usually adopted.

Withdrawal of Shares

In case any shares are withdrawn during the quarter the holders are given the amount they have paid in and interest for the equated time at 3 or 4 %. Thus, if fifty shares in the third series are withdrawn on December 1, 1918, the amount paid in is \$8.50 per share or \$425, the interest at 3% for the equated time of nine months is \$9.56, and the total amount paid is \$434.56. The book value of this stock is \$9.75 per share or \$487.50. There is a book profit of \$52.94 on the transaction. No entry is made for this profit, but it is added to the cash profits of \$489.92, which makes the divisible profits for the quarter \$542.86.

Sources of Income

The profits of building and loan associations arise from interest paid by the borrowers and also from fees, fines and premiums.

The fees are membership fees, usually 25 cents per share, paid when the stock is originally subscribed. Sometimes these fees are capitalized in the series in which they are paid, making the value per share 25 cents more. Otherwise they are treated as a general profit, as in our example. In some associations a transfer fee is paid when stock changes hands.

The fines are levied on stockholders who are delinquent in their monthly payments, usually 5 cents per share on stock of non-borrowers (investment stock), and 10 cents per share on stock of borrowers, for each month the delinquency continues.

Premiums

The premiums are either payable monthly or are deducted from the loan when made. They arise from the bidding for loans by those desiring to borrow. When the association is in possession of loanable funds, bids are invited. The person offering to pay the highest premium for the money receives the loan, if his

real estate security is found to be adequate. On the cash premium plan, 1% of the premium is paid monthly in cash. On the deducted premium plan, the whole premium is deducted from the loan in advance and only the net amount is paid.

If the loanable funds are $2,000 and the cash premium bid is 25%, the borrower makes his note for $2,000 and his monthly payments are: Dues $10, interest at 6% $10, and premium amounting to 1% of $500 premium or $5, making a total cash payment of $25.

For the same amount of money on the deducted premium plan, the bid may be for $2,500 at a premium of 20%. The borrower receives $2,000 cash, but his note is made for $2,500 and his monthly payments are: Dues $12.50 and interest $12.50, a total cash payment of $25. The premium of $500 is credited to unearned premiums, and at the end of each quarter profit and loss is credited and unearned premiums charged with $15.

Each borrower is obliged to subscribe for stock the par value of which is equal to the face of his loan. When the dues paid on this stock and the profits credited to it cause it to be worth par, it is paid off by cancelling the loan. No payments are ever applied to the loan, which remains in full force until cancelled by the matured stock.

It will be noted that in both the serial plans the treatment is entirely on the basis of the several series being fully paid to date. If stock in any series is delinquent, it still receives its share of the profit, the only offset being the small fine, which in the older series is much less than the earnings for the quarter.

Individual Plan

To meet this objection to the serial plan, some associations have adopted the individual plan, which really makes each stockholder a series by himself. The capital is then the actual amount paid in plus accumulated profits. The procedure as to earning capital and profits is otherwise the same as in the serial plan.

Dayton, or Ohio Plan

One great objection to the plans hitherto considered, or to any modification of them, is that they place the borrowers in an uncertain position with regard to the time at which their loans will be cancelled, and consequently with regard to the amount of principal and interest they will be called upon to pay. Their loans are not cancelled until their stock matures. If the association proves to be very prosperous, the profits will be large and the stock will mature in a comparatively short time. On the other hand, in a poorly managed association, it may take a long time to mature the stock, and in the meantime the borrowers will have to continue the payment of their instalments and interest, making their loans much more expensive than they have anticipated.

To meet this objection various forms of the "Dayton Plan" have been adopted, all of which agree in this—that the borrower is given a definite contract. His loan is to be repaid in a fixed number of months by the regular payment of a definite amount, which will cover both principal and interest. The calculation of the amount to be paid is based on the same principle as the one already shown for liquidating a church debt. The amount applicable to the principal is credited to the loan each month by the association, as the borrower does not have to own any stock. The loanable funds are obtained from the instalments paid by investing stockholders, with whom the accounts are kept on the individual plan.

This system is really that of a co-operative savings bank, and is the fairest method of all. When properly managed, it is very successful.

The two following problems illustrate principles dealt with in this chapter.

Problem 1

A building and loan association published the following statement on December 31, 1918:

Age and Condition of Shares

Series	Date of Issue	Number of Shares	Paid per Share	Profit per Share	Value per Share
1	January, 1917	800	$12.00	$2.56	$14.56
2	April, "	600	10.50	1.97	12.47
3	July, "	700	9.00	1.51	10.51
4	October, "	500	7.50	1.14	8.64
5	January, 1918	400	6.00	.82	6.82
6	April, "	600	4.50	.53	5.03
7	July, "	500	3.00	.26	3.26
8	October, "	600	1.50	.02	1.52

The balance sheet showed that the profits to date were $5,647.82. On March 31, 1919, the net profits from interest, premiums, fines and fees, less salary and expenses, were found to be $483.16. There had been withdrawn during the quarter 50 shares of the 4th series on which $10.75 interest had been paid, not included in the expenses previously mentioned. The 9th series was opened and 400 shares were subscribed and paid for the quarter.

Distribute the profits March 31, 1919, on the partnership plan.

Solution:

First find the equated time; which is ascertained by adding 1 to the number of months in the age of each series and dividing by 2. Next multiply the equated time by the amount paid and then by the number of shares outstanding in each series, as follows:

Series										
1	27	months old,	equated	14	mos.	×	$13.50	×	800 sh.	$151,200
2	24	" "	"	12½	"	×	12.00	×	600 "	90,000
3	21	" "	"	11	"	×	10.50	×	700 "	80,850
4	18	" "	"	9½	"	×	9.00	×	450 "	38,475
5	15	" "	"	8	"	×	7.50	×	400 "	24,000
6	12	" "	"	6½	"	×	6.00	×	600 "	23,400
7	9	" "	"	5	"	×	4.50	×	500 "	11,250
8	6	" "	"	3½	"	×	3.00	×	600 "	6,300
9	3	" "	"	2	"	×	1.50	×	400 "	1,200
										$426,675

The profits to be divided are found as follows:

Undivided profits, December 31, 1918	$5,647.82
Profits of quarter, net	483.16
	$6,130.98
Less interest paid on withdrawal	10.75
Profits to be divided	$6,120.23

The profits are divided in the following manner:

Series					
1...	2016/5689* of $6,120.23 equals	$2,168.81, or per share	$2.71
2...	1200 " " " "	1,290.96 " " "	2.15
3...	1078 " " " "	1,159.71 " " "	1.66
4...	513 " " " "	551.89 " " "	1.23
5...	320 " " " "	344.26 " " "	.86
6...	312 " " " "	335.65 " " "	.56
7...	150 " " " "	161.37 " " "	.32
8...	84 " " " "	90.37 " " "	.15
9...	16 " " " "	17.22 " " "	.04
	5,689	$6,120.24	

*151,200/426,675 = 2016/5689

Problem 2

Using the same figures as in the previous problem, distribute the profits in accordance with Dexter's rule.

Solution:

First find the capitals of the series by multiplying the value per share plus $1 by the number of shares outstanding, as follows:

Series					
1.....	800 shares by	$15.56 equals	$12,448
2.....	600 " "	13.47 "	8,082
3.....	700 " "	11.51 "	8,057
4.....	450 " "	9.64 "	4,338
5.....	400 " "	7.82 "	3,128
6.....	600 " "	6.03 "	3,618
7.....	500 " "	4.26 "	2,130
8.....	600 " "	2.52 "	1,512
9.....	400 " "	1.00 "	400
			$43,713

The profits for the quarter are	$483.16
Plus profit on withdrawn shares	46.25
Total to be divided	$529.41

The withdrawal profits are ascertained thus:

Book profit for 4th series is $1.14 per share, and for 50 shares	$57.00
Actually paid	10.75
Retained profit on withdrawal	$46.25

The divisible profit of $529.41 is a trifle more than 1.21% of the capital of $43,713. Calculated for each series it is as follows:

Series	Shares	Old Profit per Share	Old Profit of Series	New Profit of Series	Total Profit per Share	New Profit per Share
1	800	$2.56	$2,048.00	$150.62	$2,198.62	$2.75
2	600	1.97	1,182.00	97.79	1,279.79	2.13
3	700	1.51	1,057.00	97.49	1,154.49	1.65
4	450	1.14	513.00	52.49	565.49	1.26
5	400	.82	328.00	37.85	365.85	.91
6	600	.53	318.00	43.78	361.78	.60
7	500	.26	130.00	25.77	155.77	.31
8	600	.02	12.00	18.30	30.30	.05
9	400	0		4.84	4.84	.02
			$5,588.00	$528.93	$6,116.93	

The last series is usually given 2 cents as 1 cent is so very small.

With the profit per share known the statement of age and condition of shares can now be made up if asked for.

CHAPTER XIII

GOOD-WILL AND CONSOLIDATIONS

Purchasing a Business with Stock

When two or more concerns are merged, or when a business is taken over by a holding company, or when a business is bought by a corporation already in existence, the payment for the business taken over is frequently made in the stock of the corporation acquiring the business.

If the amount of stock to be given for the business is to be determined on an equitable basis, two elements in the business must be taken into consideration. These are the fair value of the net assets of the business as a going concern and the earning capacity of the business.

The fair value of the net assets may be reached by mutual agreement between the parties, or by an appraisal of the fixed assets by an appraisal company, and an estimate of the value of the active assets by an accountant.

When the value is determined, stock of a corresponding amount in face value is allotted as the price paid for the net assets.

Allocation of Net Earnings

To ascertain the earning power for which further stock should be issued, it is first necessary to establish a normal, or standard rate of earnings. This can be done only by agreement between the parties concerned. When the rate is agreed upon, it is applied to the stock already allotted for net assets. The result is the amount of net earnings to be devoted annually as dividends on the stock given in payment of the net assets.

The net earnings remaining after deducting dividends at the agreed rate on the stock issued for net assets, is the basis for

calculating the amount of stock to be allotted for the excess earning capacity of the business acquired.

The theory on which stock is allotted for excess earnings is based on the fact that one of the principal reasons for investing in a stock is that it will produce an income that is considered adequate return on the money invested. What the rate of return should be is determined by the conditions of the enterprise. An investment in a business whose conditions are stable and which may be depended upon to maintain a steady earning power, or even to increase it, yields a much lower rate of return than one in a speculative and unreliable business.

Good-Will

The name given to the value created by the excess earnings of a business is "good-will." Consideration of good-will is involved in the discussion of a stock distribution on the basis of earnings. The stock allotment is partly based on the value of the net assets acquired and partly on the good-will, or excess earning power.

Since good-will is the measure of earning capacity, it follows that only an established business can possess it. It resides usually in the reputation the business has built up in consequence of the excellence of its product, the practice of fair dealing or any other characteristics of its management giving it an advantage over its competitors. The existence of good-will cannot be proved except by a more or less prolonged experience. Therefore, when a new business is started it is false accounting to issue any stock for good-will, because its promoters expect it to earn more than the normal rate of profit. A new company, however, which has taken over an old business with a developed good-will may carry good-will among its assets and issue stock therefor.

Appraising Good-Will—Years' Purchase Method

In allotting stock the good-will may be valued in several different ways. It may be expressed as a given number of years'

purchase of the total profits. This means that the purchaser is willing to forego profits from the business for the time agreed upon. Each condition of the terms must be agreed upon between the buyer and seller. Thus, if it is agreed that the basis shall be two and one half years' purchase of the average profits for the last ten years, it will be necessary to ascertain the total profits for ten years. One tenth of this sum will be the average yearly profit, and two and one half times the average profit will be two and one half years' purchase.

If the total profit for 10 years is	$256,232.00
the average annual profit is	$ 25,623.20
and 2½ times the average is	$ 64,058.00

which is the value placed upon the good-will on the basis of two and one half years' purchase.

It is better to ascertain the average of a number of years than to take the figures for the last two and one half years, as the latter may not be normal.

This method is not as logical as allowing a certain number of years' purchase of the profits in excess of a rate agreed upon as normal. The latter calculation is the same as that given above, except that the normal profits are first deducted. Thus, if the invested capital is $200,000 and the normal rate agreed upon is 10%, the value of the good-will is figured as follows:

Average profits (as shown above)	$25,623.20
Normal profit	20,000.00
Excess profit	$ 5,623.20
8-years' purchase	$44,985.60

Capitalizing Gross Income

A third method is to capitalize the gross income. If the capital is known and the per cent of income is also known, the income is found by multiplying the capital by the rate per cent. Thus, if the capital is $200,000 and the rate of profit is

15%, the income is $30,000. On the other hand, if it is desired to ascertain what capital will produce $30,000 income if the rate per cent is 15, it is necessary to divide the income by the per cent expressed as a decimal, thus,

$30,000 divided by .15 equals $200,000.

Therefore, to capitalize a given income at a given rate per cent, the income is divided by the decimal expressing the rate.

This method, like the first, is objectionable because it does not take into consideration the fact that part of the income must be used to furnish a return on the stock already allotted for net assets. That amount of the income is in fact duplicated, and the duplication reduces the amount of the return to the concern with the largest rate of profit and increases the return to the concern with the smallest rate.

Suppose the following companies wish to combine on the basis of net assets and good-will representing gross income capitalized at 15%.

	A	B	C
Net assets and capital	$200,000	$300,000	$500,000
Average annual income	40,000	45,000	65,000
Rate of profit	20%	15%	13%

If it is proposed to form a new company with a capital of $2,000,000, the distribution of the stock is as follows:

For net assets	$200,000	$300,000	$500,000
For income capitalized at 15%	266,667	300,000	433,333
Total	$466,667	$600,000	$933,333

The total income being $150,000, the rate of profit is 7½ %, which will have to be distributed as follows:

A receives 7½ % of	$466,667	$35,000
B " " " "	600,000	45,000
C " " " "	933,333	70,000
Total	$2,000,000	$150,000

A would thus receive $5,000 less per year than when operating alone and will certainly object to the plan. B would not be affected because he is already receiving 15%, while C would gain the $5,000 lost by A.

Therefore, a fourth plan should be adopted by which the duplication of profits will be avoided. After allowing for a fixed rate of return on the capital issued for net assets to each of the merging concerns, the amount of profits necessary to provide the dividend at that rate is deducted from the total profits previously earned by the concern and the capitalization of the remaining profits is made by employing the same rate. This amounts to fixing the rate desired on the new stock and issuing stock to each concern sufficient to earn the profits previously enjoyed. If 8% is fixed upon as the rate desired, the first allotment of stock for net assets is the same as before—A $200,000, B $300,000, and C $500,000. The remaining profits are then calculated as follows:

	A	B	C	Total
Original profits	$ 40,000	$ 45,000	$ 65,000	$ 150,000
8% on net assets	16,000	24,000	40,000	80,000
Remainder to be capitalized	$ 24,000	$ 21,000	$ 25,000	$ 70,000
Capitalizing at 8%	$300,000	$262,500	$312,500	$ 875,000
Capital previously allotted	200,000	300,000	500,000	1,000,000
Total capital	$500,000	$562,500	$812,500	$1,875,000
Income at 8%	$ 40,000	$ 45,000	$ 65,000	$ 150,000

In this way each concern will receive the same earnings after the merger as before, and none will suffer.

Issue of Two Classes of Stock

We have thus far considered the issue of only one kind of stock. This necessitates the use of only one rate of return on the stock

issued. A more usual procedure when a combination of this kind is effected is the issuance of two classes of stock, preferred and common. It is an almost universal practice to issue preferred stock for the net assets and common stock for the good-will, or excess earnings.

The objection to calculating the stock given for good-will on the basis of the total earnings is the same as in the third plan. If the preferred stock is issued at 6% and the good-will is capitalized on the total profits at 20%, the common stock issued for good-will is as follows:

A	$200,000
B	225,000
C	325,000
Total	$750,000

As it takes $60,000 to pay the preferred dividend, there will be left $90,000 for the common stock, which will allow a 12% dividend. The profits will therefore be divided as follows:

A	$200,000 preferred at 6%	$12,000	
	200,000 common at 12%	24,000	$36,000
B	300,000 preferred at 6%	$18,000	
	225,000 common at 12%	27,000	45,000
C	500,000 preferred at 6%	$30,000	
	325,000 common at 12%	39,000	69,000
Total			$150,000

In this way A makes a loss and C a gain of $4,000.

The proper method is to issue the common by capitalization of the excess profits after the preferred dividends have been provided for, thus:

	A	B	C
Original profits	$40,000	$45,000	$65,000
Preferred dividends 6%	12,000	18,000	30,000
Excess remaining	$28,000	$27,000	$35,000
Capitalized at 20%	$140,000	$135,000	$175,000

It is evident that dividends of 6% on the preferred and 20% on the common stock will give the same return as formerly.

After the preferred dividends are deducted it does not make any difference what basis is adopted for capitalizing, because whatever rate is used the relative proportion remains the same. Thus, if the excess is capitalized at four years' purchase the resultant common stock is as follows: A $112,000, B $108,000, and C $140,000, a total of $360,000, on which the $90,000 remaining profits will provide a dividend rate of 25%, and the distribution will be thus:

A for preferred	$12,000	
" common, 25% of $112,000	28,000	$40,000
B " preferred	$18,000	
" common, 25% of $108,000	27,000	45,000
C " preferred	$30,000	
" common, 25% of $140,000	35,000	65,000

The practice of starting a new business with a capital stock of a par value greater than the total of the net assets is often indulged in. In such a case a debit must be made to some account, in order to balance the books. The account debited is often called good-will. This is altogether wrong, since, as we have seen, good-will is a matter of growth and cannot be possessed by a business which has had no time to develop it.

CHAPTER XIV

FOREIGN EXCHANGE

Conversion of Foreign Coinage

In order to provide a basis for the comparative values of United States and foreign coinage, the Director of the United States Mint periodically estimates the values of foreign coins, which are proclaimed by the Secretary of the Treasury. The values thus fixed are called the mint pars of exchange, and represent the intrinsic or bullion values of foreign coins in terms of United States money. A table of values recently proclaimed appears in the Appendix.

To reduce a value in foreign coinage to United States money, multiply the value of the coin in United States money by the number of coins.

Illustration

What will a draft for £410/19/6 cost in dollars at the mint par of exchange?

SOLUTION: There are two ways to calculate this:

1. Reduce shillings and pence to decimals of the pound, thus:

	£410.
19 shillings are 19/20 of a pound	.95
6 pence are 6/240 or 1/40 of a pound	.025
	£410.975

£410.975 multiplied by $4.8665 is $2,000.01.

2. Reduce pence, shillings and pounds to value in dollars, thus:

6 pence are 1/40 of $4.8665	$ .12
19 shillings are 19/20 of $4.8665	4.62
410 pounds at $4.8665	1,995.27
£410/19/6 at $4.8665	$2,000.01

Reverse Conversion

To reduce a value in United States coinage to a foreign money value, divide the value in United States money by the value of the foreign unit.

Illustration

What is the value in pounds of $2,000 at the mint par of exchange?

SOLUTION: There are again two ways to calculate this, which are as follows:

1. 2,000 ÷ 4.8665 = 410.973, number of pounds
   .973 of 20 (shillings in the pound) = 19.46
   .46 of 12 (pence in the shilling) = 5.52
   Hence $2,000 = £410/19/6.

2. To find the number of pounds:

```
                4.8665 | 2,000.00 | 410£
                       1,946.60
                       ---------
                          53.400
                          48.665
                          ------
Remainder                  4.7350
Reduce to shillings           20
                       ---------
                 48665 | 947000 | 19 s
                       48665
                       -----
                        460350
                        437985
                        ------
Remainder               22365
Reduce to pence            12
                       ---------
                 48665 | 268380 | 5.52 d
                       243325
                       ------
                        250550
                        243325
                        ------
                         72250
```

Result £ 410/19/6.

In this particular case there is a much shorter method, which is as follows:

£410/19/6 is the same as £411 minus 6d, or £411 minus 1/40 of £1. We can, therefore, obtain the result by performing the following subtraction:

£411/00/0 × $4.8665	$2,000.13
— 6 or 1/40 of $4.8665	—.12
£410/19/6	$2,000.01

Current rates of exchange vary from the unit par rate, because the supply and demand for foreign drafts are affected by the balance of trade between countries. Transactions involving foreign exchange are made at current instead of mint par rates.

Exchange is quoted at the current value in cents and conversions are made as illustrated in the conversions at mint rates.

Illustration

What is the cost of a 300 franc draft on Paris, purchased at 8.49?

SOLUTION: $.0849 × 300 = $25.47

A United States merchant sends $5,000 to Paris when the rate is 8.49. What is the value in francs?

SOLUTION: 5,000 ÷ .0849 = 58,892.8, the number of francs.

Dealing in Foreign Exchange

There are two methods of recording transactions in foreign exchange. When the first method is used, the charges and credits are entered in two values, domestic and foreign, the conversion of each item being made at the rate applying to the transaction. When the account is closed the balance in foreign money is valued by converting either at cost or current rate, and the profit or loss on exchange is ascertained.

The other method attempts to ascertain the profit or loss on each item, by computing the difference between the conversion value at par and current rates. Both methods are illustrated below:

Illustration

A banking concern deals in foreign exchange and the following are the transactions with a London correspondent for one month:

DEBITS

Sept.	1	Remittance, 30-day bill	£400	at	$4.86
"	10	" sight bill	£100/10	"	4.87
"	15	" " "	£200/0/6	"	4.86¾

CREDITS

Sept.	2	Draft, sight	£300	at	$4.87½
"	12	" "	£200/12/5	"	4.87
"	20	Cable	£100	"	4.88

1. Ascertain the profit or loss in the account for the month.
2. State the balance of the account at the end of the month in foreign and domestic currency, the current rate of sterling exchange for cable transfers being $4.89.

SOLUTION 1: Although the problem states that the current rate of exchange for cable transfers is $4.89, it seems incorrect to inventory the balance of the account at this rate, since the rate is not only higher than cost but higher than any of the prices at which sales have been made. On the theory that the first items purchased are the first sold we find that the first two remittances are offset by the first two credits. Applying this theory, the balance of the account must have cost $4.86¾, the rate applying to the remittance of September 15. Setting up the account and valuing the balance at $4.86¾, we have the following:

ACCOUNT WITH LONDON CORRESPONDENT

DEBITS

			Foreign	Rate	Domestic
Sept.	1,	Remittance, 30-day bill.....	£400/00/0	$4.86	$1,944.00
"	10,	" sight bill.....	100/10/0	4.87	489.44
"	15,	" " ".....	200/00/6	4.86¾	973.62
"	30,	Profit			6.74
			£700/10/6		$3,413.80
Oct.	1,	Balance—Inventory.....	£99/18/1	$4.86¾	$486.28

CREDITS

Sept. 2, Draft, sight	£300/00/0	$4.87½	$1,462.50
" 12, " "	200/12/5	4.87	977.02
" 20, Cable	100/00/0	4.88	488.00
" 30, Balance	99/18/1	4.86¾	486.28
	£700/10/6		$3,413.80

While $6.74 is shown as profit it must be remembered that the rate of 4.86¾ is used because the current rate for checks is unknown, that interest has been disregarded in the calculation, and that the profit as shown is nominal only and may be increased or decreased when realized by a sale.

If the balance were inventoried at $4.89, its domestic value would be $488.53, and the profit would be $8.99.

It would be correct to value the balance at the current buying rate on September 30, if that were known. This is the usual practice.

In calculating the dollar value of the balance, the shortest way is to take £99/18/1 as 1 s 11 d less than £100/00/0, as follows:

1 shilling is 1/20 of $4.8675 or		$0.24337
11 pence is 1/12 less than 1 shilling, that is,	$.24337	
	−.02027	.22310
		$.47
£100	$486.75	
− 1/11	−.47	
£ 99/18/1	$486.28	

SOLUTION 2: The above is the usual method of keeping a foreign exchange account in this country. In Canada each transaction as recorded in the buying and selling registers is compared with the par of exchange and an entry made debiting or crediting Exchange, as the case may be. The par of sterling is $4.8665 or $4.86⅔. The registers, in the example taken, would show the following:

BOUGHT

DATE	TIME	STERLING	RATE	PAID	PAR	DR. EX.	CR. EX.
Sept. 1	30 d	£400/00/0	$4.86	$1,944.00	$1,946.67	$	$2.67
10	st.	100/10/0	4.87	489.44	489.10	.34	
15	"	200/00/6	4.86¾	973.62	973.45	.17	
		£700/10/6			$3,409.22	$.51	$2.67

			Sold			
Sept. 2	st.	£300/00/0	$4.87½	$1,462.50	$1,460.00	$2.50
12	"	200/12/5	4.87	977.02	976.35	.67
20	ca.	100/00/0	4.88	488.00	486.67	1.33
30	balance	99/18/1			486.20	
		£700/10/6			$3,409.22	$4.50

The ledger would contain an account with the London correspondent charged and credited with the entries in the Par columns, and an account with Exchange, showing a debit of $0.51 and credits of $2.67 and $4.50. The sterling column is a memorandum inventory account only. The account with Exchange would indicate a profit of $6.66, the difference from the profit of $6.74 shown by the United States method being caused by inventorying the balance at par instead of at cost. Of course, the postings would be made in daily totals in a bank in which a daily trial balance is taken off.

The balance may be entered at cost of $4.86¾, or $486.28 in the sold column and 8 cents in the credit exchange column, which will make the profit $6.74, the same as by the other method.

Average Date of Current Account

The principles already explained for determining the average date of an account apply when the account is kept in a foreign coinage. The following is an illustration.

Illustration

A of London, in current account with B of New York, engages an accountant to prepare a statement, to be mailed to B, based on the following data:

1914
Debits:

May 12	£750	
May 30	117	
June 12	340	
July 1	150	
Total debits		£1,357

Credits:		
June 10	£500	
June 30	300	
Total credits		800
Balance		£ 557

Find the average due date of the account and the interest at 5% to July 1, taking 365 days to the year.

SOLUTION:

Focal date, April 30—

DEBITS

May 12	£ 750	12 days	£ 9,000
" 30	117	30 "	3,510
June 12	340	43 "	14,620
July 1	150	62 "	9,300
	£1,357		£36,430

CREDITS

June 10	£ 500	41 days	£20,500
" 30	300	61 "	18,300
	£ 800		38,800
Balances	£ 557 Debit		£2,370 Credit

The amount, 2,370 divided by 557 equals 4. But since the balance of the account is a debit while the balance of the products is a credit, the average date is four days backward from April 30, or April 26.

From April 26 to July 1 is 66 days. Interest on £557 for 66 days at 5% may be calculated in either of the following ways:

$$1.\quad £557 \times .05 \times \frac{66}{365} = £5.57 \times \frac{66}{73} = £5.036$$

£1 × .036 = 20 s × .036 = .72 s

1 s × .72 = 12 d × .72 = 8.64 d

Total interest £5/0/9

2. At 6%	£5.57	is int. for 60 days on 360 days basis
" 6%	.557	" " " 6 " " " " "
	£6.127	" " " 66 " " " " "
" 1%	1.021	" " " " " " " " "
" 5%	£5.106	" " " " " " " " "
Reducing to 365 day basis	.07	Subtract 1/73
	£5.036	which is reduced to £5/0/9 as above

The decimal .036 can also be reduced as follows:

£1 = 240 d
£ .036 = 240 × .036, or 9 d

Conversion of Foreign Branch Accounts

The following example shows how foreign branch account balances are converted to domestic values, and how the home office and foreign branch accounts are consolidated in the periodical statements.

Illustration

A New York corporation builds a plant and establishes a branch in Liverpool, England. At the expiration of its fiscal period, a trial balance is forwarded to the New York office, as follows:

Plant	£250,000	
Accounts receivable	187,500	
Expenses	25,000	
Inventory (end of fiscal period)	50,000	
Remittance account	150,000	
Cash	12,500	
Accounts payable		£ 87,500
Income from sales		250,000
New York office		337,500
	£675,000	£675,000

A trial balance of the New York books on the same date is as follows:

Capital stock		$2,500,000.00
Patents	$1,500,000.00	
Liverpool account	1,640,250.00	
Remittance account		729,281.25
Expenses at New York	25,000.00	
Cash	64,031.25	
	$3,229,281.25	$3,229,281.25

The remittance account is composed of four 60-day drafts on Liverpool for £37,300 each, which were sold in New York at $4.85½, $4.86, $4.86½ and $4.86¾ respectively.

Prepare a balance sheet of the New York books after closing and a statement of assets and liabilities of the Liverpool branch consolidated with the New York books. Close the books at the rate of exchange on the last day of the fiscal period, which is $4.87¼, conversion of remittances to be made at the average rate for the four bills.

SOLUTION: Since the fixed assets are not subject to revaluation at current exchange rates, the plant cost must be taken out of the New York office account, which is subdivided to show indebtedness to the main office for fixed assets and for current assets. Since the cost of the fixed assets is not given, we must assume that it is at the same rate as the rest of the New York account. Dividing the Liverpool account balance of $1,640,250 by the New York office account balance of £337,500 we obtain the exchange rate of 4.86, at which rate the plant has a value in United States money of $1,215,000. The reciprocal accounts are then subdivided as follows:

On Liverpool books:

New York office—plant	£250,000
" " " —current account	87,500

On New York books:

Liverpool account—plant	$1,215,000.00
" current account	425,250.00

The next step is to close the Liverpool books into the New York office current account as follows:

NEW YORK OFFICE CURRENT ACCOUNT

Expenses	£ 25,000	Balance	£ 87,500
Remittance account	150,000	Income from sales	250,000
Balance	162,500		
	£337,500		£337,500
		Balance	£162,500

The balance sheet of the Liverpool branch is now as follows: (Plant valued at cost, $4.86; current assets and liabilities valued at current exchange rate of $4.87¼).

ASSETS

Plant	$4.86	£250,000	$1,215,000.00
Accounts receivable	4.87¼	187,500	913,593.75
Inventory	4.87¼	50,000	243,625.00
Cash	4.87¼	12,500	60,906.25
		£500,000	$2,433,125.00

LIABILITIES

Accounts payable	$4.87¼	£ 87,500	$ 426,343.75
New York office—plant account	4.86	250,000	1,215,000.00
New York office—current account	4.87¼	162,500	791,781.25
		£500,000	$2,433,125.00

On the New York books, the Liverpool current account is credited with the remittances, and a charge is made to the account (offset by a credit to profit and loss) sufficient to leave the account with a balance of $791,781.25, the balance of the net current assets in Liverpool at $4.87¼. It is not necessary to find the average rate of the remittance account since the whole account is used. It would be found, by dividing $729,281.25 by 150,000, to be $4.86 3/16. The figures to be used are then found by multiplying £150,000 by $4.86 3/16, which gives $729,281.25, an amount which we already have.

LIVERPOOL CURRENT ACCOUNT

Balance	$ 425,250.00	Remittances	$ 729,281.25
Profit	1,095,812.50	Balance	791,781.25
	$1,521,062.50		$1,521,062.50
Balance	$ 791,781.25		

The New York profit and loss account will appear as follows:

PROFIT AND LOSS

New York expenses	$ 25,000.00	Liverpool Branch	$1,095,812.50
Net profit to surplus	1,070,812.50		
	$1,095,812.50		$1,095,812.50

This profit is subject to a further charge for reduction of the patents account.

The New York office trial balance is as follows:

Capital stock		$2,500,000.00
Surplus		1,070,812.50
Patents	$1,500,000.00	
Liverpool plant account	1,215,000.00	
Liverpool current account	791,781.25	
Cash	64,031.25	
	$3,570,812.50	$3,570,812.50

The balance sheet is as follows:

ASSETS

Fixed assets:			
Patents		$1,500,000.00	
Liverpool Plant		1,215,000.00	$2,715,000.00
Current assets:			
Accounts receivable, Liverpool		$ 913,593.75	
Inventory		243,625.00	
Cash—Liverpool	$60,906.25		
Cash—New York	64,031.25	124,937.50	1,282,156.25
			$3,997,156.25

LIABILITIES

Capital stock	$2,500,000.00	
Surplus	1,070,812.50	$3,570,812.50
Accounts payable, Liverpool		426,343.75
		$3,997,156.25

The Liverpool branch profit of $1,095,912.50 is made up as follows:

Income from sales	£250,000	
Less expenses	25,000	
Net profit on sales	£225,000, at $4.87¼	$1,096,312.50
Increase of current exchange rate ($4.87¼) over original valuation ($4.86) of opening balance of current account:		
£87,500 at $4.87¼	$426,343.75	
£87,500 at $4.86	425,250.00	1,093.75
		$1,097,406.25
Less exchange loss on remittances:		
£150,000 at $4.87¼ (current rate)	$730,875.00	
£150,000 at $4.861875 (average conversion rate)	729,281.25	1,593.75
Net profit		$1,095,812.50

Even in this explanation it is not necessary to use the average conversion rate.

CHAPTER XV

LOGARITHMS

Use

Logarithms are a special compilation of figures, used to reduce the labor of multiplication, division, computation of powers and extraction of roots. After explaining the method of employing logarithms, their theory will be briefly discussed.

Every number has its corresponding logarithm—but a knowledge of the method of determining the logarithms of various numbers is not essential, as they may be obtained from prepared tables. The logarithms of certain numbers are stated below for use in illustrations. For a more complete table, see Appendix.

TABLE OF LOGARITHMS OF SELECTED NUMBERS

NUMBER	LOGARITHM
2	.30103
3	.47712
4	.60206
5	.69897
6	.77815
7	.84510
8	.90309
9	.95424
10	1.00000
12	1.07918
15	1.17609
16	1.20412
30	1.47712
100	2.00000
160	2.20412
300	2.47712
1,000	3.00000
1,600	3.20412
3,000	3.47712

Multiplication

Multiplication is accomplished by the addition of logarithms, as follows:

1. In the table, find the logarithms of the numbers to be multiplied.
2. Add the logarithms; the sum is the logarithm of the desired product.
3. In the table, find the number corresponding to the logarithm obtained in 2.

The following examples, though more easily performed without logarithms, will illustrate the steps of the process:

Example

$2 \times 3 = ?$

SOLUTION:

	NUMBER	LOGARITHM	
1. Table shows	2	.30103	
" "	3	.47712	
2. Sum of logarithms		.77815	log of product

3. Table shows .77815 is log of 6, the product

Example

$2 \times 3 \times 2 = ?$

SOLUTION:

	NUMBER	LOGARITHM	
1. Table shows	2	.30103	
" "	3	.47712	
" "	2	.30103	
2. Sum of logarithms		1.07918	log of product

3. Table shows 1.07918 is log of 12, the product

Without going beyond the selected table, the following multiplications may be performed by using logarithms:

$5 \times 6 = ?$ $4 \times 3 = ?$ $3 \times 2 \times 5 = ?$

Division

Division is accomplished by the subtraction of logarithms as follows:

1. In the table find the logarithms of the dividend and the divisor.
2. Subtract the logarithm of the divisor from the logarithm of the dividend; the difference is the logarithm of the desired quotient.
3. In the table, find the number corresponding to the logarithm obtained in 2.

Example

$6 \div 2 = ?$

SOLUTION:

	NUMBER	LOGARITHM	
1. Table shows:			
Dividend.....	6	.77815	
Divisor.....	2	.30103	
2. Difference of logarithms.....		.47712	log of quotient

3. Table shows .47712 is log of 3, the quotient

The following divisions may be performed by logarithms with the aid of the selected table:

$8 \div 2$ $10 \div 5$ $16 \div 4$ $300 \div 30$

Calculating Powers

A power of a number is the product obtained by using that number repeatedly as a factor. The square of a number is the

second power; the cube, the third power, etc. The exponent of the power indicates the number of times the factor is to be used.

For instance 3^2 means the second power of 3; or 3 squared; the number used as a factor is 3. The superscript number ^{2} is the exponent of the power, showing that 3 is to be used twice as a factor.

Thus, $3^2 = 3 \times 3 = 9$

A number is raised to a power by logarithms as follows:

1. In the table find the logarithm of the number to be raised to a power.
2. Multiply the logarithm by the exponent of the power; the product is the logarithm of the power.
3. In the table find the number corresponding to the logarithm obtained in 2.

Example 1

$3^2 = ?$

SOLUTION:

	NUMBER	LOGARITHM	
1. Table shows.....	3	.47712	
2. Multiply by exponent of power.....		2	
		.95424	log of power

3. Table shows .95424 is log of 9, the second power of 3.

Example 2

$2^4 = ?$

SOLUTION:

	NUMBER	LOGARITHM	
1. Table shows.....	2	.30103	
2. Multiply by exponent of power.....		4	
		1.20412	log of power

3. Table shows 1.20412 is log of 16, the fourth power of 2.

Roots

A root is a number which, used repeatedly as a factor, will produce a given power.

The square root of 9 is 3, because 3 used twice as a factor produces 9.

The fourth root of 16 is 2, because 2 used 4 times as a factor produces 16.

The root of a number is expressed by writing the number under the radical sign; the index figure indicates the root to be extracted.

The square root of 9 is expressed thus:

$$\sqrt[2]{9}$$

The superscript 2 is the index of the root.

Any root of a number may be extracted by logarithms as follows:

1. In the table find the logarithm of the number, the root of which is to be extracted.
2. Divide the logarithm by the index of the root; the quotient is the logarithm of the desired root.
3. In the table find the number corresponding to the logarithm obtained in 2.

Example 1

$$\sqrt[2]{9} = ?$$

SOLUTION:

1. Table shows log of 9 to be .95424
2. Divide by index of root: $.95424 \div 2 = .47712$, log of root
3. Table shows .47712 is log of 3, the square root of 9

Example 2

$$\sqrt[4]{16} = ?$$

SOLUTION:

1. Table shows log of 16 to be 1.20412
2. Divide by index of root: $1.20412 \div 4 = .30103$, log of root
3. Table shows .30103 is log of 2, the fourth root of 16

Nature of Logarithms

Logarithms are exponents of the base 10. The logarithm of any number shows how many times 10 must be used as a factor to produce that number. This is easily seen in the case of numbers which are exact powers of 10. The selected table shows the following:

NUMBER	LOGARITHM	
10	1.00000	
100	2.00000	$(10 \times 10 = 100)$
1000	3.00000	$(10 \times 10 \times 10 = 1000)$

Numbers which are the exact powers of 10 have logarithms which are whole numbers.

Most numbers which are not exact powers of 10 have logarithms which are mixed numbers, composed of two parts: an integer, and a decimal fraction.

The following are taken from the selected table:

NUMBER	LOGARITHM
10	1.00000
12	1.07918
15	1.17609
16	1.20412
30	1.47712
100	2.00000
160	2.20412
1,000	3.00000
1,600	3.20412

The Characteristic and the Mantissa

The two parts of a logarithm are: the *characteristic*, which is an integer, and the *mantissa*, which is the decimal part.

The mantissa is determined by the figures of the number; (initial and final zeroes in the number do not affect the mantissa). Thus the logarithms of the numbers 16, 160, 1600, etc., all have the same mantissa, .20412.

The characteristic indicates the exponent of the largest power of 10 contained in the number.

To illustrate:

Number	Largest Power of 10 Contained in Number	Exponent of Power	Logarithm
16.	10	1	1.20412
160.	100	2	2.20412
1600.	1000	3	3.20412
16000.	10000	4	4.20412

It will be noted that the characteristic of the logarithm is always one less than the number of figures at the left of the decimal point in the number.

Number	Figures at Left of Decimal Point	Characteristic	Logarithm
16	2	1	1.20412
160	3	2	2.20412
1600	4	3	3.20412
16000	5	4	4.20412

Tables of Logarithms

Referring to the table of logarithms in the Appendix it will be noted that only the mantissas are given; in forming logarithms the mantissa is taken from the table, and the characteristic is determined by the location of the decimal point in the number. The following lines are taken from the table in the Appendix:

No.	0	1	2	3	4	5	6	7	8	9
160	20412	20439	20466	20493	20520	20548	20575	20602	20629	20656
161	20683	20710	20737	20763	20790	20817	20844	20871	20898	20925

The heavy figures at the left and at the top indicate the sequence of figures in the number.

The smaller numbers in the body of the table indicate the mantissas. For instance:

Line 160, Column 0, shows 20412, which is the mantissa of 16, 160, 1,600, 16,000, etc.

Line 160, Column 4, shows 20520, which is the mantissa of 16.04, 160.4, 1,604, 16,040, etc.

Line 161, Column 9, shows 20925, which is the mantissa of 16.19, 161.9, 1,619, 16,190, etc.

Applying the method explained in the preceding section and using the sequence of figures 1,619 as an illustration, the characteristic is ascertained and prefixed as follows:

Number	Figures at Left of Decimal Point	Characteristic	Logarithm
16,190	5	4	4.20925
1,619	4	3	3.20925
161.9	3	2	2.20925
16.19	2	1	1.20925

Characteristics of Logarithms of Numbers between 1 and 10

From the preceding illustration it will be noted that as the decimal point is moved one place to the left the characteristic decreases by 1.

The table is continued in the following:

Number	Figures at Left of Decimal Point	Characteristic		Logarithm
16.19	2	1		1.20925
1.619	1	0		0.20925
			or	.20925

Thus it is seen that the characteristic is 0 when the number lies between 1 and 10, which is in accordance with the general rule that the characteristic is one less than the number of figures at the left of the decimal point.

Examples

Determine the characteristics and state the logarithms of the following numbers:

25,000	2.5	492
2,500	37,010	7,635
250	370.1	200
25	3.701	4,005

Use of the Characteristic in Pointing Off Results

If the characteristic of a logarithm is one less than the number of figures at the left of the decimal point in the number, then the figures at the left of the decimal point in the number must be one more than the characteristic.

Referring to the brief table on page 141, 20790 is the mantissa of the number 1,614; the characteristic preceding this mantissa would determine the location of the decimal point. To illustrate:

Logarithm	Characteristic	Places at Left of Decimal Point	Number
4.20790	4	5	16,140.
3.20790	3	4	1,614.
2.20790	2	3	161.4
1.20790	1	2	16.14
.20790	0	1	1.614

The following examples are given to illustrate the relation between the characteristic and the position of the decimal point.

Illustration

$$24 \times 3.6 = ?$$

Solution (Refer to table for mantissas):

1. Determine logarithms:

log 24:	mantissa	.38021
	characteristic	1
	logarithm	1.38021
log 3.6:	mantissa	.55630
	characteristic	0
	logarithm	0.55630

2. Add logarithms:

1.38021	log 24
0.55630	" 3.6
1.93651	log of product

3. Determine number corresponding to the logarithm 1.93651.

Mantissa .93651 is found in the table thus:

No.	0
864	93651

and represents sequence 8640.

Characteristic 1 indicates that there are to be two places at the left of the decimal point in the product, 86.4.

Illustration

$$9.46 \div 4.3 = ?$$

SOLUTION:

1.	log 9.46:	mantissa	.97589
		characteristic	0
		logarithm	0.97589
	log 4.3:		
		mantissa	.63347
		characteristic	0
		logarithm	0.63347
2.	0.97589 log of dividend		
	0.63347 " " divisor		
	0.34242 " " quotient		
3.	.34242 is mantissa of figures 2200		
	Characteristic 0; one figure at left of decimal point; quotient 2.2		

Examples

Find the value of the following:

1. 23×15
2. 35×2.5
3. 11.4×7
4. $392 \div 8$
5. $952 \div 28$
6. $9.38 \div 1.4$
7. 8^3
8. 4^4
9. 1.2^2
10. $\sqrt[6]{729}$
11. $\sqrt[3]{64}$
12. $\sqrt[2]{5.29}$

Logarithms of Numbers Smaller than 1

The following numbers with their logarithms have been used to show that moving the decimal point in a number one place to the left reduces by 1 the characteristic of the logarithm.

Number	Figures at Left of Decimal Point	Characteristic	Logarithm
16,190.	5	4	4.20925
1,619.	4	3	3.20925
161.9	3	2	2.20925
16.19	2	1	1.20925
1.619	1	0	0.20925

If the decimal point is again moved one place to the left, we have:

.1619	0	1	1.20925

In other words, if the number is less than 1, the characteristic of the logarithm is a negative quantity. The minus sign is placed above the negative characteristic to show that it applies to the characteristic only, the mantissa being a positive quantity. The logarithm $\bar{1}.20925$ may also be written 9.20925–10.

The negative characteristic is one greater than the number of zeroes between the decimal point and the first significant figure in the number.

In the following, the table of numbers smaller than 1 and their logarithms is continued:

Number	Zeros Between Decimal Point and First Significant Figure	Negative Characteristic	Logarithm
.1619	0	−1	$\bar{1}.20925$ or $9.20925-10$
.01619	1	−2	$\bar{2}.20925$ or $8.20925-10$
.001619	2	−3	$\bar{3}.20925$ or $7.20925-10$
etc.			

Examples

Determine the characteristics by inspection; refer to the table for the mantissas; and state the logarithms of the following numbers:

.346	.0346	.00346	.000346	
.0201	.0031	.29	.0101	.000003

Use of Negative Characteristic in Pointing Off Answers

When the logarithm obtained by a computation has a negative characteristic, it indicates that the corresponding number is smaller than 1; that is, it is a decimal fraction. Point off the number by placing between the decimal point and the first significant figure one fewer zeros than the number of the characteristic thus:

Logarithm	Negative Characteristic	Zeros	Number
$\bar{3}.20925$	−3	2	.001619
$\bar{2}.20925$	−2	1	.01619
$\bar{1}.20925$	−1	0	.1619

Examples

Determine the numbers corresponding to the following logarithms:

$\bar{2}.98005$ $\bar{3}.88969$ $\bar{1}.99782$ 0.71642

Computing with Logarithms Having Negative Characteristics

The following are illustrations of the use of logarithms with negative characteristics, in multiplication, division, raising to powers, and extraction of roots.

Multiplication

Illustration

$.021 \times .000451 = ?$

SOLUTION:

NUMBER	LOGARITHM
.021	$\bar{2}.32222$ or $8.32222-10$
.000451	$\bar{4}.65418$ or $6.65418-10$
Sum of logs	$\bar{6}.97640$ or $14.97640-20$

.97640 is the mantissa of the figures 9471
Characteristic $\bar{6}$; five zeroes in answer; product, .000009471

Division

Illustration 1

$.0341 \div .31 = ?$

SOLUTION:

	NUMBER	LOGARITHM
Dividend	.0341	$\bar{2}.53275$
Divisor	.31	1.49136
Logarithm of quotient		$\bar{1}.04139$

.04139 is the mantissa of the figures 11
Characteristic $\bar{1}$; no zeroes; quotient .11

Illustration 2

$.341 \div .00031 = ?$

SOLUTION:

	NUMBER	LOGARITHM
Dividend	.341	$\bar{1}.53275$
Divisor	.00031	4.49136
Logarithm of quotient		3.04139

.04139 is mantissa of 11
Characteristic 3 is positive; point off four places from the left; quotient 1,100

Illustration 3

$$34.1 \div .31 = ?$$

SOLUTION:

	NUMBER	LOGARITHM
Dividend	34.1	1.53275
Divisor	.031	$\bar{2}.49136$
Logarithm of quotient		3.04139

(Note: subtracting -2 is the same as adding $+2$)
.04139 is the mantissa of 11
Characteristic 3; put decimal point four places from left; quotient 1,100

Powers

Illustration

$$.034^2 = ?$$

SOLUTION:

	NUMBER	LOGARITHM
	.034	$\bar{2}.53148$
Multiply by exponent		2
Logarithm of power		$\bar{3}.06296$

Note: the $\bar{3}$ characteristic is the result of the following:

	CHARACTERISTIC	MANTISSA
Logarithm	$\bar{2}$	.53148
Multiply by	2	2
	$\bar{4}$	1.06296

The difference between $\bar{4}$ and 1 is $\bar{3}$, as above
.06296 is mantissa of 1156
Characteristic $\bar{3}$; two zeroes required; hence $.034^2 = .001156$

Roots

Illustration

$$\sqrt[3]{.000729} = ?$$

SOLUTION:

NUMBER	LOGARITHM
.000729	$\bar{4}.86273$

This logarithm is to be divided by 3; but if -4 is divided by 3, the quotient is -1, and $-333 +$ to carry. This negative remainder causes confusion in the division of the positive mantissa. Hence the procedure below is followed:

	NUMBER	LOGARITHM
	.000729	$\bar{4}.86273$ or $6.86273 - 10$
Add and subtract 20*.....		$20. \quad - 20$
Divide by 3.....		$3\,)\,26.86273 - 30$
		$8.95424 - 10$
		or $\bar{2}.95424$

.95424 is the mantissa of 9
Characteristic $\bar{2}$; one zero; hence $\sqrt[3]{.000729} = .09$
*20 is used to obtain the -30, which is divisible by 3

Examples

Compute the following:

1. $.031 \times .3$
2. $.0045 \times .00021$
3. $.0036 \times 2.2$
4. $.000011 \times 56000$
5. $.0023 \times 42000$
6. $.006 \times 6.5$
7. $.038 \times .0003$
8. $504 \div 24$
9. $.368 \div 23$
10. $.00351 \div .027$
11. $390 \div .15$
12. $.0994 \div 71$
13. 28^2
14. 1.3^2
15. $.9^3$
16. $.003^6$
17. $\sqrt[4]{625}$
18. $\sqrt[4]{.0625}$
19. $\sqrt[3]{.000343}$
20. $4^3 \times \sqrt[4]{.0016}$

Determining Mantissas by Interpolation

Tables of logarithms are limited in scope and do not contain the mantissas of all possible numbers. For numbers beyond the scope of the table, the mantissa may be approximated by interpolation.

Illustration

What is the logarithm of 18565?

Solution:	Number	Mantissa
The table shows.....	18570	.26881
" " ".....	18560	.26858
Difference.....	10	.00023

If a difference of 10 in the numbers causes a change of .00023 in the mantissa, a difference of 5 will cause a change of approximately $^5/_{10}$ of .00023 in the mantissa.

Now, .5 of .00023 = .000115

Then	.26858,	mantissa of 18560
plus	.000115	
sum	.268695,	mantissa of 18565 (approximate)

Logarithm, 4.268695

Examples

By interpolation determine the logarithms of the following numbers:

3642	1209	47629	758263
25.69	3.479	.004682	32.0046

Determining Numbers by Interpolation

When a computation results in a mantissa not to be found in the table, the number corresponding to the mantissa may be determined approximately by interpolation.

Illustration

$15^4 = ?$

SOLUTION:

	NUMBER	LOGARITHM
	15	1.17609
Multiply by exponent of power.....		4
Logarithm of 15^4.....		4.70436

The mantissas larger and smaller than .70436 shown in the table are:

	NUMBER	MANTISSA
	5063	.70441
	5062	.70432
Differences	1	.00009
The mantissa of 15^4 is		.70436
" " " 5,062 is		.70432
Difference		.00004

If an increase of .00009 in the mantissa represents an increase of 1 in the number, an increase of .00004 in the mantissa represents an increase of approximately 4/9 of 1, or .444 + in the number.

Hence	.70432	Mantissa of 5062
add interpolation for	.00004	444 +
Result	.70436 mantissa of	5062444 +

4.70436 is the logarithm of 50624.44 or 15^4
Exact answer is 50,625.

Examples

What are the approximate numbers corresponding to the following logarithms?

2.76354	4.94965
3.42345	$\bar{2}$.4166572
1.82629	$\bar{3}$.57927

Illustration

What is the value of 1856^2?

SOLUTION:

The logarithm of 1,856, is	3.26858
Multiply by exponent of power	2
Logarithm of 1856^2	6.53716

Table shows:	NUMBER	MANTISSA
Larger mantissa	3445	.53719
Smaller "	3444	.53706
Differences	1	.00013
Mantissa of	1856^2	.53716
" "	3444	.53706
Difference		.00010

10/13 of 1 = .769 approximately

Hence	3444	.53706
Plus	769	.00010
Sum	3444769	.53716

Then 3444769 is the number represented by 6.537148

1856^2 is accurately	3444736
Approximated by logarithms	3444769
Error	33

If the problem had been 18.56^2

The accurate power would be	344.4736
Approximate power (by logs)	344.4769
Error	.0033

This example shows that fairly accurate results may be obtained by interpolation, but absolutely accurate work requires tables extended to show the mantissas of all numbers required by the problem.

Examples

What are the values of:

1. 24×18
2. 360×800
3. 32.6×3
4. $.27 \times .71$
5. $864 \times .00321$
6. 2462×3278
7. $.00964 \times 3.425$
8. $34400 \div 43$
9. $9280 \div 232$
10. $11000 \div 12$
11. $5893 \div 830$
12. $754.314 \div 6234$
13. 17^2
14. 1225^2
15. 2.6^{10}
16. $.0346^8$
17. $.001294^3$
18. $\sqrt[3]{372}$
19. $\sqrt[4]{704}$
20. $\sqrt[4]{20736}$
21. $\sqrt[2]{.00183331 6}$
22. $\sqrt[5]{.0000000032}$
23. $\sqrt[2]{\frac{768}{12}}$
24. $\sqrt[7]{\frac{81^2}{3}}$

CHAPTER XVI

SIMPLE AND COMPOUND INTEREST

Simple Interest—Methods of Calculating

Interest is the income or expense arising from the use of money or credit and the increase in investment or indebtedness resulting therefrom.

When only the original indebtedness bears interest, the principles of simple interest apply; when the interest increase of the indebtedness also bears interest, the principles of compound interest apply.

There are a number of methods of computing simple interest. Those given below are among the easiest. These methods are based upon a rate of 6%, since that is a common rate and is besides a factor of 12, the number of months in a year, and of 30, the number of days in a month. Adjustments may be easily made for other rates. The methods described are also based on 360 days to a year. Corrections may be made to adjust the result to a basis of 365 days.

A rate of 6% (.06) per year is equivalent to a rate of 1% (.01) for 2 months or 60 days (one sixth of a year). As .01 of a number is computed by moving the decimal point two places to the left, the interest on any principal at 6% for 2 months or 60 days may be computed by moving the decimal point two places to the left. Since 6 days are one tenth of 60 days, interest for 6 days may be computed by moving the decimal point three places to the left. And since 600 days are ten times 60 days, interest at 6% for 600 days may be computed by moving the decimal point one place to the left.

Thus, to find the interest at 6% on any principal for

6 days, point off in the principal 3 decimal places to the left
60 " " " " " " 2 " " " " " "
600 " " " " " " 1 " place " " "
6000 " the interest is the same as the principal

For instance, the interest at 6% on $8,245.75 for

6 days is $8.24575 or $8.25
60 " " $82.4575 or $82.46
600 " " $824.575 or $824.58
6,000 " " $8,245.75

The time is frequently a number of days which may be separated into parts, each of which is a fraction or multiple of 6, 60, 600, or 6,000. In such cases the interest may be computed by finding the total of the interest for the component time parts.

Illustration

Find the interest on $726.32 for 88 days.

SOLUTION:

$ 7.2632 = interest for 60 days
2.4211 = " " 20 " (⅓ of 60)
.72632 = " " 6 "
.24211 = " " 2 " (⅓ of 6)
$10.65273 = " " 88 "

If the days are not convenient fractions or multiplies of 6, 60, etc., the principal may be such a fraction or multiple, in which case principal and time may be interchanged thus:

Illustration

Find the interest on $3,600 for 37 days.

SOLUTION: This is equivalent to interest on $37 for 3600 days. Then $3.70 (pointing off one place) the interest for 600 days, multiplied by 6 equals $22.20 the interest on $37 for 3,600 days, or the interest on $3,600 for 37 days.

When neither of these methods can be utilized, the procedure is as follows:

Point off three places in the principal—Result: Interest for six days.
Multiply by number of days—Result: Interest for six times the number of days.
Divide by 6—Result: Interest for given number of days.

Illustration

Find the interest on $827.52 for 119 days at 6%.

SOLUTION:

	$ .82752	interest for 6 days
	119	
	$98.47488	" " 6 × 119 days
$98.47488 ÷ 6 =	$16.41248	" " 119 days

Rates Other than 6%

To calculate simple interest when the rate is other than 6%, the interest at 6% is found and the result is adjusted to the given rate.

Illustration

What is the interest on $827.52 for 119 days at 4½%?

SOLUTION: The interest at 6% has already been computed being $16.41248. Since 4½% is ¾ of 6%, the interest at 4½% is found as follows:

$16.41248	interest at	6 %
4.10312	"	" 1½% (¼ of 6%)
$12.30936	"	" 4½%

365 Day Basis

To adjust to a basis of exact interest, 365 days to a year, the interest is computed on a 360 day basis as explained, and the result is decreased $^1/_{73}$ of itself. The 5 days difference is $^1/_{73}$ of 365 days, and since interest is a computation which may be expressed

$$\frac{\text{Principal} \times \text{Rate} \times \text{Days}}{\text{Number of days in Year}}$$

the use of 360 (a denominator which is $^1/_{73}$ too small) will result in the interest's being $^1/_{73}$ too large.

Illustration

What is the exact interest on $827.52 at 4½% for 119 days?

SOLUTION: The interest at 4½% on a 360 day basis has already been computed, being $12.30936
Deduct $^1/_{73}$ of $12.30936 ..... .16862
Exact interest on 365 day basis $12.14074

Partial Payments

There are two methods of calculating interest on a debt on which partial payments are made. The one in common use among business men and known as the Merchants' Rule, consists in calculating the interest on the principal sum from its date to the date of settlement and a similiar calculation on the partial payments, the difference between the total interest on the respective sides being the net interest due. On small sums and for short periods this is accurate enough for all practical purposes.

The other, called the United States Rule, gives precedence to the interest due at the time of each payment, and requires that each payment shall be first applied to the liquidation of the interest then due, only the remainder after the interest is deducted being applicable to the reduction of the principal. If the payment is not equal to the interest then due, it is applied in reducing the interest, but the excess interest is not added to the principal,

as this would be compounding it, but it is carried down and added to the interest to be deducted from the next payment. This method is made legal by statute in nearly every state.

An example of the two methods will show the difference between them, 30 days to the month being used for convenience.

Illustration

Jan. 1.	Original amount	$600	March 1, payment		$200
			May 1, "		200
			June 1, "		100

Interest is to be charged at 6%. Required—amount due July 1

SOLUTION:

By the Merchants' Rule:

Interest on $600 for 6 months is		$18.00
" " 200 " 4 " "	$4.00	
" " 200 " 2 " "	2.00	
" " 100 " 1 " "	.50	6.50
Interest due July 1		$11.50
Unpaid principal		100.00
Total due July 1		$111.50

By the United States Rule:

Original debt		$600.00
Payment March 1	$200.00	
Less interest on $600 for 2 months	6.00	194.00
		$406.00
Payment May 1	$200.00	
Less interest on $406 for 2 months	4.06	195.94
		$210.06
Payment June 1	$100.00	
Less interest on $210.06 for 1 month	1.05	98.95
		$111.11
Interest on $111.11 for 1 month		.56
Total due July 1		$111.67

While the difference of 17 cents in the results obtained by the two methods is negligible, it is otherwise when the amounts are large and the time is long. The following example is taken from the accounts of an estate that was not settled for thirty years. It was virtually as follows:

Illustration

An amount of $36,000 was due one of the heirs of an estate, who was to receive 6% interest until paid.

January 1, 1886, legacy		$36,000
" 1, 1891, payment		12,000
" 1, 1896, "		12,000
" 1, 1906, "		12,000
" 1, 1911, "		19,000

The administrator, using the commercial rule, agreed to settle the account as of January 1, 1916, as follows:

Original amount		$ 36,000
Interest for 30 years at 6%		64,800
		$100,800
Less total cash paid	$55,000	
Interest on $12,000 for 25 years	18,000	
Interest on $12,000 for 20 years	14,400	
Interest on $12,000 for 10 years	7,200	
Interest of $19,000 for 5 years	5,700	100,300
Amount due January 1, 1916		$500

The heir refused the settlement and made his claim under the law of the state (Illinois) as follows:

Original amount		$36,000.00
January 1, 1891, payment	$12,000.00	
Interest 5 years on $36,000.00	10,800.00	1,200.00
		$34,800.00
January 1, 1896, payment	$12,000.00	
Interest 5 years on $34,800.00	10,440.00	1,560.00
		$33,240.00

January 1, 1906, payment.....		$12,000.00	
Interest 10 years on $33,240.00..		19,944.00	
Interest carried forward.....		$ 7,944.00	
January 1, 1911, payment.....		$19,000.00	
Interest 5 years on $33,240.00...	$9,972.00		
Interest brought forward.....	7,944.00	17,916.00	1,084.00
			$32,156.00
January 1, 1916, interest 5 years on $32,156.00.....			9,646.80
Amount due January 1, 1916..			$41,802.80

It can readily be seen that a little knowledge of correct principles was a valuable asset to the heir.

Compound Interest

The principles of interest are involved in the computations which an accountant may be required to make in connection with such matters as bond premium and discount, leasehold premiums, depreciation and sinking funds. These calculations involve not only compound interest and the amount of a given principal at compound interest, but also the more complex problems of annuities.

All scientific computations of interest on an indebtedness or an investment extending over more than one period of time must be based on compound interest. This is because the indebtedness or investment increases with the lapse of time. If only simple interest is charged, interest is earned only on the original investment instead of on the investment for each period. For instance, if $100 is invested at 6%, the investment at the beginning of the first year is $100. It increases $6 during the first year and amounts at the end of the first year to $106.

Since the investment has now increased to $106, interest

should be earned thereon. If simple interest only is charged, $6 will be earned during the second year on an investment of $106. This is at the rate of only 5.66 +%.

Scientific computations of interest are based on the supposition that the increase arising from interest is re-invested. This is equitable in theory, for if the interest of one period increases the investment at the close of that period, the investor should, during the next period, earn interest on the increased investment. To maintain the agreed interest rate during a series of periods, computations must be made on the basis of periodical compounding.

Symbols

The following standard symbols by Sprague and Perrine, will be used throughout this book:

$1 = $1, £1, or any other unit of value
i = the rate of interest for a single period
n = an indefinite number of periods
a = the amount of $1 for a given time at a given rate
I = the compound interest on $1 for a given time at a given rate
p = the present worth of $1 for a given time at a given rate
D = the discount on $1 for a given time at a given rate
r = $(1 + i)$, the periodic ratio of increase[1]

Amount of Principal

In finding the amount of a given principal at compound interest for a given number of years at a given rate per year compounded annually, it is customary to compute the amount of $1 for the given time at the given rate, and multiply this result by the number of dollars in the principal.

When $i = 6\%$ or .06, the accumulation of the amount may be computed by either of the following methods:

---

[1] C. E. Sprague and L. L. Perrine, The Accountancy of Investment, 1914.

First Method			Second Method		
	Dollars	End of		Dollars	Symbols
Principal.....	1.00			1.00	1
Interest on $1.00.....	.06		Multiply by	1.06	$(1+i)$
	1.06	1 year		1.06	$(1+i)$
" " 1.06.....	.0636		" "	1.06	$(1+i)$
	1.1236	2 "		1.1236	$(1+i)^2$
" " 1.1236.....	.067416		" "	1.06	$(1+i)$
	1.191016	3 "		1.191016	$(1+i)^3$
" " 1.191016.....	.071461		" "	1.06	$(1+i)$
	1.262477	4 "		1.262477	$(1+i)^4$

The ratio of increase is, in symbols, $(1+i)$, and in figures 1.06. The investment at the beginning of each year must, therefore, be multiplied by $(1+i)$ to obtain the investment, or amount, at the end of that year. Hence 1 invested at the rate i becomes at the end of

1 year.....$(1+i)$
2 "..... $(1+i)^2$
3 "..... $(1+i)^3$
4 "..... $(1+i)^4$
n "..... $(1+i)^n$

Thus the formula is obtained

$$a = (1+i)^n$$

If it is desired to find the amount of 1 at 5% for 20 years, compounded annually, the formula becomes

$$a = (1.05)^{20}$$

Frequency of Compounding

In the preceding illustration the interest was compounded annually. But the period of compounding may be shorter than one year, with the result that the compounding occurs with greater frequency than once a year. In such cases the formula stated above still applies. Although compounding may occur

semiannually, quarterly, monthly or even daily, the rate is usually stated as a certain per cent per year, but i in the formula is the annual rate divided by the number representing the periods in a year. For instance, if the rate is 6% per year, i under various conditions would be as follows:

Frequency of Compounding		Value of i
Annually		.06
Semiannually	(.06 ÷ 2)	.03
Quarterly	(.06 ÷ 4)	.015
Monthly	(.06 ÷ 12)	.005
Daily	(.06 ÷ 365)	$\frac{.06}{365}$

The number of periods, represented in the formula by n, will be the number of years times the number of periods per year. For instance n, under various conditions, is as follows:

Frequency of Compounding	Value of n		
	One Year	Two Years	Three Years
Annually	1	2	3
Semiannually	2	4	6
Quarterly	4	8	12
Monthly	12	24	36
Daily	365	730	1095

Illustration

What is the amount of $1 at 6% interest for 20 years, compounded quarterly?

Solution:

$i = .06 \div 4 = .015$, the rate for one period
$n = 20 \times 4 = 80$, the number of periods
$a = (1 + i)^n$ or $(1.015)^{80}$, which amounts to 3.29066279

Determining the Amount

Interest tables show the amount of $1 at various rates for various periods. Such a table appears in the Appendix of this

book. When a table is not available, the amount may be computed with a *table of logarithms*. When neither an interest table nor a table of logarithms is available, it is necessary to compute the amount by repeated multiplication, thus:

$$a = (1.03)^8 = ?$$

$$1.03 \times 1.03 = 1.0609 = (1.03)^2$$

$$1.0609 \times 1.03 = 1.092727 = (1.03)^3$$

$$1.092727 \times 1.03 = 1.12550881 = (1.03)^4$$

At this point the number of decimal places becomes so large as to make the computation too laborious. The number may be reduced to six places, by approximation, thus:

1.12550881 becomes 1.125509

The multiplication continued is as follows:

$$1.125509 = (1.03)^4$$

$$1.125509 \times 1.03 = 1.159274 = (1.03)^5$$

$$1.159274 \times 1.03 = 1.194052 = (1.03)^6$$

$$1.194052 \times 1.03 = 1.229874 = (1.03)^7$$

$$1.229874 \times 1.03 = 1.266770 = (1.03)^8$$

This last amount is the amount of \$1 at 6% interest for 4 years, compounded semiannually

But when the number of periods is large, as will be the case when the interest must be calculated over a long term of years, these repeated multiplications become irksome. It is possible, however, to cut down on the number of operations necessary. A short method may be utilized which should be readily understood if it is remembered that $(1+i)^n$ means that $(1+i)$ is used n times as a factor. If we know the value of $(1+i)^2$ we have the amount obtained by using $(1+i)$ twice as a factor. By squaring this amount we get $(1+i)^4$, which is the product resulting from the use of $(1+i)$ four times as a factor. Squaring this amount in turn gives us $(1+i)^8$, the product obtained by using $(1+i)$ eight times as a factor.

By the application of this principle, a very material reduction may be effected in the number of multiplications, as is made clear in the example given below:

$$\begin{array}{rl} 1.03 & = (1+i) \\ 1.03 & = (1+i) \\ \hline 1.0609 & = (1+i)^2 \\ 1.0609 & = (1+i)^2 \\ \hline 1.125509 & = (1+i)^4 \\ 1.125509 & = (1+i)^4 \\ \hline 1.266770 & = (1+i)^8 \end{array}$$

The quantity 1.03^{10} can be determined in the following manner:

$$\begin{array}{ll} 1.26677 & (1+i)^8 \\ 1.0609 & (1+i)^2 \\ \hline 1.343916 & (1+i)^{10} \end{array}$$

The general application of this principle may be made clearer by another illustration.

Illustration

Required—the value of $(1 + i)^{30}$:

Solution:

$$
\begin{array}{l}
(1+i) \\
(1+i) \\
\hline
(1+i)^{2} \\
(1+i)^{2} \\
\hline
(1+i)^{4} \\
(1+i)^{4} \\
\hline
(1+i)^{8} \\
(1+i)^{8} \\
\hline
(1+i)^{16} \\
(1+i)^{8} \\
\hline
(1+i)^{24} \\
(1+i)^{4} \\
\hline
(1+i)^{28} \\
(1+i)^{2} \\
\hline
(1+i)^{30}
\end{array}
$$

Thus, only seven multiplications are required to obtain the 30th power.

The 80th power could be obtained thus:

$$
\begin{array}{l}
(1+i)^{16} \text{ (obtained by four multiplications)} \\
(1+i)^{16} \\
\hline
(1+i)^{32} \\
(1+i)^{32} \\
\hline
(1+i)^{64} \\
(1+i)^{16} \\
\hline
(1+i)^{80}
\end{array}
$$

The same principle may be utilized in the use of compound interest tables which do not extend to the desired number of periods.

Illustration

Given a table of 40 periods, required—the value of $(1 + i)^{95}$.

SOLUTION:

$(1 + i)^{40}$ shown by table
$(1 + i)^{40}$ " " "

$(1 + i)^{80}$
$(1 + i)^{15}$ " " "

$(1 + i)^{95}$

Determining Interest

The amount of $1 at compound interest is composed of two elements: the original investment of $1 and the accumulated interest. Hence, to find the compound interest, apply the following formula:

$$I = a - 1$$
$$\text{or } I = (1 + i)^n - 1$$

If $1.266770 is the amount of $1 at 6% interest, compounded semiannually for four years, $1.000000 is the original investment, and $.266770 is the compound interest.

Determining Present Worth

The present worth of a sum due at a fixed future date is a smaller sum which with interest will amount to the future sum. For instance, the present value of $1 due in one year at 6% is a sum, smaller than $1, which with interest at 6% for one year will amount to $1. Representing the present value of this $1 by p,

$$p \times 1.06 = \$1$$

or in symbols, $p \times (1 + i) = 1$ when the dollar is due in one year.

Since $p \times (1 + i) = 1$,

it follows that $1 \div (1 + i) = p$

(Since $p \times 1.06 = 1$, it follows that $1 \div 1.06 = .943396$.)

If the dollar is due two periods hence,

$$p \times (1 + i)^2 = 1$$

and it follows that $1 \div (1 + i)^2 = p$

If the dollar is due three periods hence

$$p \times (1 + i)^3 = 1$$

and it follows that $1 \div (1 + i)^3 = p$

If the dollar is due n periods hence

$$p \times (1 + i)^n = 1$$

and it follows that $1 \div (1 + i)^n = p$

But since $(1 + i)^n = a$

the formula $p = 1 \div (1 + i)^n$

may also be stated $p = 1 \div a$

In determining the present value at 6% of $1 due in four years, interest compounded semiannually, several methods are available.

1. It may be possible to refer to a table of present values, similar to the one in the Appendix.
2. If a table of present values is not available, the present value may be computed by using the formula

$$p = 1 \div a$$

It will first be necessary to obtain the value of a. When interest at 6% per annum is compounded semiannually, $i = .03$; with semiannual compounding for four years, $n = 8$. Then, $a = (1.03)^8$. The value of a may be found by any of the methods previously explained; it is 1.266770. Then

$\$1 \div 1.266770 = \$.789409$, the present value

3. Instead of dividing $1 by $(1.03)^8$, i.e., by 1.266770, the same result can be obtained by dividing by 1.03 eight times, using 1 as the first dividend, each succeeding dividend being the quotient resulting from the preceding division, thus

$1.000000 ÷ 1.03 = $.970874, present value of $1 due in 1 period
.970874 ÷ 1.03 = .942596, " " " 1 " " 2 periods
.942596 ÷ 1.03 = .915142, " " " 1 " " 3 "
.915142 ÷ 1.03 = .888487, " " " 1 " " 4 "
.888487 ÷ 1.03 = .862609, " " " 1 " " 5 "
.862609 ÷ 1.03 = .837484, " " " 1 " " 6 "
.837484 ÷ 1.03 = .813092, " " " 1 " " 7 "
.813092 ÷ 1.03 = .789409, " " " 1 " " 8 "

Determining the Compound Discount

As shown in the preceding section, one would, in exchange for a promise to pay $1 a given number of periods hence, loan the present value of $1. Present values for various numbers of periods, and the discount earned, are shown below (discounted at 3% per period):

Due Periods Hence	Amount Loaned Present Value	Discount Earned $(1-p)$
1	$.970874	$.029126
2	.942596	.057404
3	.915142	.084858
4	.888487	.111513
8	.789409	.210591

The discount for two or more periods is called compound discount. Compound discount may be computed by the formula

$$D = 1 - p$$

It may also be computed by the formula

$$D = I \div a$$

This formula requires explanation. Compound discount is really compound interest, deducted in advance; but it is the compound interest on the money actually loaned.

For instance, if \$.789409 is loaned on a promise to pay \$1 eight periods hence at 3%, the discount \$.210591 is 3% compound interest on \$.789409, the principal actually loaned. This fact can be demonstrated thus:

The amount of \$1 at 3% compound interest for eight periods is \$1.266770. Hence the compound interest on \$1 is \$.266770 and the compound interest on \$.789409 is the result in the following multiplication:

	.789409	p
Multiplied by	.266770	i
Product	.210591	d

Since the compound discount (D) is really the compound interest (I) on the actual loan (p),

$$D = I \times p$$

And since

$$p = 1 \div a,$$

we can substitute $(1 \div a)$ for p, and the formula becomes

$$D = I \times \frac{1}{a}$$

or

$$D = I \div a$$

This formula may also be explained thus: The present value of \$1 due in eight periods at 3% is \$.789409, or $\$1 \div (1.03)^8$, or $\$1 \div a$.

If the loan were 1, the compound interest would be I, or \$.266770; but since the loan is $1 \div a$, the compound discount is $I \div a$, or \$.266770 ÷ 1.266770 = \$.210590.

Summary

In this chapter the following formulas have been derived:

$$a = (1 + i)^n$$
$$I = a - 1$$
$$p = 1 \div a; \quad \text{or } 1 \div (1 + i)^n$$
$$D = 1 - p; \quad \text{or } I \div a$$

CHAPTER XVII

ANNUITIES

Definition of Annuities

A series of equal payments, due at regular intervals, is an annuity. Although the word "annuity" suggests annum and year, the interval may be any period, as a month, quarter or half-year.

Symbols

In the discussion of annuities the following symbols will be employed:

A = the amount of an annuity of \$1 for a given time at a given rate

P = the present value of an annuity of \$1 for a given time at a given rate

Amount of an Annuity

Let us assume that a contract provides for payments as in the following:

Example

January 1, 1915	\$100
" 1, 1916	100
" 1, 1917	100
" 1, 1918	100
Total	\$400

Required—the accumulated value of this annuity at January 1, 1918, interest at 5%.

SOLUTION: It is assumed that each payment is put at interest, in which case the amount of each payment computed separately, is as follows:

Payment Made	Periods at Interest	Payment	Amount	
			Symbol	$
January 1, 1915	3	$100	$(1+i)^3$	115.7625
" 1, 1916	2	100	$(1+i)^2$	110.2500
" 1, 1917	1	100	$(1+i)$	105.0000
" 1, 1918	0	100	1	100.0000
Amount of the annuity				431.0125

Although the amount of an annuity may be computed by determining the amount of each payment, it is unnecessary to resort to this labor, as the following short method may be used:

To find the amount of an annuity of $1 for a given number of periods at a given rate, divide the compound interest on $1 for the number of periods at the given rate, by the interest rate.

Or, in symbols, $A = I \div i$

Applying this formula to the illustration above, I, the compound interest on $1 for four periods at 5%, is shown by an interest table to be $.215506; then

$ .215506 ÷ .05 = $4.31012, amount of an annuity of $1
$4.31012 × 100 = $431.012, amount of an annuity of $100

This formula requires explanation. Let us suppose that $1 is loaned on January 1, 1914, the contract requiring payment of simple interest annually at 5% as follows:

Date	Interest
January 1, 1915	$.05
" 1, 1916	.05
" 1, 1917	.05
" 1, 1918	.05

These payments, being equal in amount and made at regular intervals, constitute an annuity of $.05 per year. If put at 5% compound interest, the amount of each payment computed separately would be as follows:

Payment Made	Periods at Interest	Payment	Amount	
			Symbols	$
January 1, 1915	3	$.05	$(1+i)^3$	.057881
" 1, 1916	2	.05	$(1+i)^2$	.055125
" 1, 1917	1	.05	$(1+i)$	.052500
" 1, 1918	0	.05	1	.050000
Totals		$.20		.215506

The total of the annuity payments is $.20, the simple interest on $1 for four years; and the amount of the annuity, $.215506, is the compound interest on $1 for four years, or I. Hence, compound interest is merely the amount of an annuity. If the compound interest $.215506 is the amount at 5% of four annual payments of 5 cents,

$.215506 ÷ 5 = $.0431012, the amount at 5% of four annual payments of $.01

or $.215506 ÷ .05 = $4.31012, the amount at 5% of four annual payments of $1.

or in symbols $I \div i = A$

Illustration

Required the amount of an annuity of $25 per month for five years at 6% per annum, interest compounded monthly.

Solution:

12 (payments per year) × 5 = 60, number of terms

6% ÷ 12 = ½% or .005, the rate

Now, $I = a - 1$

And $a = (1.005)^{60} = 1.348849$

Then $A = I \div i$

$= .348849 \div .005$

$= 69.7698$, amount of an annuity of $1

$69.7698 × 25 = $1,744.25, amount of an annuity of $25

Sinking Fund Contribution

The preceding section developed a method of determining the amount which results from the accumulation of a known annuity. It is the purpose of this section to discuss the converse problem, of finding an unknown annuity which will produce a required amount.

This problem finds application among accountants in computing the periodical contribution necessary to accumulate a required sinking fund. Let us assume that a sinking fund of $100,000 is to be accumulated in five years by equal instalments made at the end of each year; what is the required contribution, assuming 4% interest compounded annually?

The annual contributions constitute an annuity and the accumulated fund is the amount of the annuity. We shall first find what fund would be accumulated by a contribution of $1, applying the formula

$$A = I \div i$$

To find I:

	1.04	
Multiplied by	1.04	
	1.0816	$= (1 + i)^2$
" "	1.0816	
	1.16985856	$= (1 + i)^4$
" "	1.04	
	1.2166529024	$= (1 + i)^5$
Deduct	1.	
	.2166529024	$= I$, compound interest on $1 for five periods at 4%

To find A:

.2166529024 ÷ .04 = 5.41632256, amount of an annuity of $1

Since annual contributions of $1 will produce a fund of $5.41632256, to find the contributions necessary to produce a fund of $100,000:

$100,000 ÷ 5.41632256 = $18,462.71, required contribution

The computation was performed without using an interest table. If tables are available, the work may be materially decreased.

For instance, a table of amounts of $1 per annum shows the amount of $1 for five periods at 4% to be $5.416323. Only one computation is necessary:

$100,000 ÷ 5.416323 = $18,462.71

If only a compound interest table is available, it will show the amount of $1 in five periods at 4% to be $1.216653. The following computations will be necessary:

$.216653 ÷ .04 = $5.416325, amount of annuity of $1
$100,000 ÷ 5.416325 = $18,462.71

The following tabulation of sinking fund accumulations shows the accumulation of the fund from the two elements, annual contributions and interest:

End of Year	Contribution	Interest	Total Fund
1	$18,462.71		$18,462.71
2	18,462.71	$ 738.51	37,663.93
3	18,462.71	1,506.56	57,633.20
4	18,462.71	2,305.33	78,401.24
5	18,462.71	3,136.05	100,000.00
Totals	$92,313.55	$7,686.45	

Present Worth of an Annuity

Let us assume that a contract, made on January 1, 1914, provides for the following payments:

January 1, 1915	$100
" 1, 1916	100
" 1, 1917	100
" 1, 1918	100
Total	$400

We are to find the present value of this series of payments, or annuity, on January 1, 1914, discounted at 5%.

The present value of each payment, computed separately, is as follows:

Payment Due	Periods Hence	Payment	Present Value	
			Symbol	$
January 1, 1915	1	$100	$1 \div (1 + i)$	95.2381
" 1, 1916	2	100	$1 \div (1 + i)^2$	90.7029
" 1, 1917	3	100	$1 \div (1 + i)^3$	86.3838
" 1, 1918	4	100	$1 \div (1 + i)^4$	82.2702
Totals		$400		354.5950

The present value of each payment, and the present value of the annuity, may be computed thus:

$100	÷ 1.05 =	$ 95.2381	present value of	$100	due in	1	year
95.2381	÷ 1.05 =	90.7029,	" " "	$100	" "	2	years
90.7029	÷ 1.05 =	86.3838,	" " "	$100	" "	3	"
86.3838	÷ 1.05 =	82.2702,	" " "	$100	" "	4	"
		$354.5950,	present value of the annuity.				

When interest tables are not available, this method is satisfactory; but with interest tables available a short method may be used which is similar to the short method of computing the amount of an annuity.

To find the present value of an annuity of $1 for a given

number of periods at a given rate, divide the compound discount on $1 for the given number of periods by the interest rate. Or in symbols,

$$P = D \div i$$

Applying this formula to the illustration above, the present value of $1 due four periods hence at 5%, is shown by an interest table to be $.822702;

D, the compound discount, is $.177298.
$.177298 ÷ .05 = $3.54596, the present value of an annuity of $1
$3.54595 × 100 = $354.596, " " " " " " " $100

This formula also requires explanation. Let us suppose that a loan of $1 is made on January 1, 1914, the contract requiring the payment of simple interest annually at 5% as follows:

Date	Interest
January 1, 1915	$.05
" 1, 1916	.05
" 1, 1917	.05
" 1, 1918	.05

These payments, being equal in amount, and made at regular intervals, constitute an annuity of five cents per year. If discounted at 5%, the present value of each payment on January 1, 1914, computed separately, is as follows:

Payment Due	Periods Hence	Payment	Present Value	
			Symbols	$
January 1, 1915	1	$.05	$1 \div (1 + i)$	.0476191
" 1, 1916	2	.05	$1 \div (1 + i)^2$	.0453514
" 1, 1917	3	.05	$1 \div (1 + i)^3$	.0431919
" 1, 1918	4	.05	$1 \div (1 + i)^4$	.0411351
Present value of the annuity				.1772975

Now, instead of paying five cents interest each year, and paying the $1 at maturity, the present value ($.1772975) of the four interest payments might be paid, or deducted, in advance, thus:

$1.0000000 payment to be made at end of four years
.1772975 present value of interest payments

$ .8227025 present value of $1

In other words, the simple interest on $1 at 5% for four years is an annuity of five cents; and $.1772975, the present value of these interest payments, is the present value at 5% of an annuity of four five cents payments; and $.1772975 is also the compound discount on $1 due four periods hence at 5%. Therefore the compound discount on $1, due in four periods at 5%, is the present value of an annuity of five cents in four periods at 5%.

If the compound discount $.1772975 is the present value at 5% of four annual payments of 5 cents, then

$.1772975 ÷ 5 = $.0354595, the present value at 5% of four annual payments of 1 cent

or $.1772975 ÷ .05 = $3.54595, the present value at 5% of four annual payments of $1

or in symbols $D \div i = P$

Illustration

Required the present value of an annuity of $25 per month for five years at 6% per annum, compounded monthly.

SOLUTION: 12 (payments per year) × 5 = 60, number of periods
6% ÷ 12 = ½% or .005, the rate

$D = 1 - p$
and $p = 1 \div (1.005)^{60} = .741372$
Then $D = 1 - .741372 = .258628$, the compound discount on $1 due in sixty periods at ½ %

and

$$D \div i = P$$

$\$ .258628 \div .005 = \$ 51.7256$, present value of an annuity of \$1

$\$51.7256 \times 25 = \$1{,}293.14$, the present value of an annuity of \$25

Rent of an Annuity

The preceding section, containing an explanation of the method of determining the present worth of a known annuity, developed the formula, $P = D \div i$. Each periodical instalment of an annuity is known as the rent of an annuity. This section will deal with the question of determining what rent will be produced by a known present worth.

It has been shown that the present value of an annuity of \$1 for four periods at 5% is \$3.54595.

Since a present value of \$3.54595 will produce four annual rents of \$1 at 5%, a present value of \$1 will produce four annual rents of $\frac{1}{3.54595}$ of \$1, or \$.282012.

Therefore, to find the rent of an annuity with a present value of \$1 for a given number of periods at a given rate:

1. Find the present value of an annuity of \$1 for the given time and rate
2. Divide \$1 by this present value of an annuity of \$1

Or expressed in symbols,

$$R = 1 \div P$$

Illustration

A man invests \$3,000 in an annuity to be repaid to him in five annual instalments, interest computed at 6% annually. What annual rent will be produced?

SOLUTION: Formula—$R = 1 \div P$

The present value of $1 due five periods hence at 6% may be computed thus:

$$1 \div (1.06)^5$$

or it may be determined from interest tables. The compound interest table shows

$$1.06^5 = 1.338226$$

then

$1 \div 1.338226 = .747258$, the present value of $1 due in five periods

$1 - .747258 = .252742$, the compound discount on $1 due in five periods

$.252742 \div .06 = 4.21236$, the present value of an annuity of $1 for five periods at 6%

then

$R = 1 \div 4.21236 = .237396$, rent produced by present worth of $1

$\$.237396 \times 3{,}000 = \$712.19$, rent produced by present worth of $3,000

The following schedule shows the reduction of the investment due to the excess of the periodical payments over the interest earned on the decreasing investment.

Original investment	$3,000.00
Interest 1st year, 6% of $3,000	180.00
	$3,180.00
Deduct 1st rent	712.19
Balance, end of 1st year	$2,467.81
Interest 2d year, 6% of $2,467.81	148.07
	$2,615.88
Deduct 2d rent	712.19
Balance, end of 2d year	$1,903.69
Interest 3d year, 6% of $1,903.69	114.22
	$2,017.91
Deduct 3d rent	712.19

Balance, end of 3d year	$1,305.72
Interest 4th year, 6% of $1,305.72	78.34
	$1,384.06
Deduct 4th rent	712.19
Balance, end of 4th year	$671.87
Interest, 5th year, 6% of $671.87	40.31
	$712.18
Deduct 5th rent	712.18
	$ .00

The schedule could also be arranged as follows, showing that each payment is composed of two parts:

1. Interest on the diminishing principal
2. Repayment of the principal

Original investment		$3,000.00
First rent:		
Interest on $3,000 at 6%	$180.00	
Repayment of principal	532.19	532.19
	$712.19	$2,467.81
Second rent:		
Interest on $2,467.81 at 6%	$148.07	
Repayment of principal	564.12	564.12
	$712.19	$1,903.69
Third rent:		
Interest on $1,903.69 at 6%	$114.22	
Repayment of principal	597.97	597.97
	$712.19	$1,305.72
Fourth rent:		
Interest on $1,305.72 at 6%	$ 78.34	
Repayment of principal	633.85	633.85
	$712.19	$ 671.87
Fifth rent:		
Interest on $671.87 at 6%	$ 40.31	
Repayment of principal	671.87	671.87
	$712.18	$ .00

Or the schedule may be shown thus:

Payment	Rent	Interest	Reduction of Investment	Diminishing Investment
				$3,000.00
1	$ 712.19	$180.00	$ 532.19	2,467.81
2	712.19	148.07	564.12	1,903.69
3	712.19	114.22	597.97	1,305.72
4	712.19	78.34	633.85	671.87
5	712.18	40.31	671.87	0
Totals	$3,560.94	$560.94	$3,000.00	

Equal Periodical Payments on Principal and Interest

When a debt together with the interest thereon is to be paid in equal periodical instalments, the principal of the debt is the present value of an annuity, and the periodical payments are rents, to be computed by the method already explained.

To illustrate, if $3,000 bearing 6% interest compounded annually, is to be paid in five annual instalments, each to include the accrued interest and a portion of the principal, $3,000 is the present value of five unknown rents. These rents were computed in the preceding illustration, being $712.19. The reduction of the debt may be tabulated thus:

Original principal		$3,000.00
First payment	$712.19	
Less 1 year's interest on $3,000	180.00	
Payment on principal		532.19
Balance of principal		$2,467.81
Second payment	$712.19	
Less 1 year's interest on $2,467.81	148.07	
Payment on principal		564.12
Balance of principal		$1,903.69
Third payment	$712.19	
Less 1 year's interest on $1,903.69	114.22	
Payment on principal		597.97

Balance of principal		$1,305.72
Fourth payment	$712.19	
Less 1 year's interest on $1,305.72	78.34	
Payment on principal		633.85
Balance of principal		$671.87
Fifth payment	$712.18	
Less 1 year's interest on $671.87	40.31	
Payment on principal		671.87
Balance of principal		$ .00

Annuities Due

In the foregoing discussion of annuities, rents, and sinking fund contributions, the formulas and methods described apply to the ordinary form of annuities, in which the payments are made at the end of the periods.

When the payments are made at the beginning of the periods, the annuity is called an annuity due. Changing the payment from the end to the beginning of the period affects the compound interest, thus changing both the amount and the present value of the annuity.

To Find the Amount of an Annuity Due

To arrive at a method of computing the amount of an annuity due, let us compare the amounts of:

1. An ordinary annuity (A) of six periods (due at end of period)
2. An annuity due (B) of five periods (due at beginning of period)

In each annuity, the interest is compounded annually at 4%.
A requires the payment of $1 on December 31 for each of six years, beginning December 31, 1911.

B requires the payment of $1 on January 1 for each of five years, beginning January 1, 1912.

It will be noted that the first five payments of annuities A and B are made on practically identical dates; but annuity A has one more payment than annuity B; hence the amount of annuity A will exceed the amount of annuity B by one payment of $1. This may be more clearly shown by the following table which shows the amount of each annuity payment and the amount of the annuity.

Annuity A			Annuity B		
Date	Payment	Amount at Dec. 31, 1916	Date	Payment	Amount at Dec. 31, 1916
Dec. 31, 1911.....	$1.00	$1.216653	Jan. 1, 1912.....	$1.00	$1.216653
" 31, 1912.....	1.00	1.169859	" 1, 1913.....	1.00	1.169859
" 31, 1913.....	1.00	1.124864	" 1, 1914.....	1.00	1.124864
" 31, 1914.....	1.00	1.081600	" 1, 1915.....	1.00	1.081600
" 31, 1915.....	1.00	1.040000	" 1, 1916.....	1.00	1.040000
" 31, 1916.....	1.00	1.00			
		$6.632976			$5.632976

Annuity B is an annuity due of five rents; annuity A is an ordinary annuity of six rents; the periodical payments and interest rate are the same in each case. The difference between the two amounts is $1, or one periodical payment.

Hence we can find the amount of annuity B (annuity due) for five periods by ascertaining the amount of an ordinary annuity (A) of the same payment and rate for six periods, and deducting $1. The amount of annuity A (ordinary) may be computed thus:

$1.265319, amount of $1 for six periods
1.00

$ .265319, compound interest for six periods

$.265319 ÷ .04 = $6.632975, amount of an ordinary annuity of six periods

To compute the amount of B, the annuity due:

	$6.632975, amount of ordinary annuity of six periods
Deduct	1.000000, one rent
	$5.632975, amount of annuity due of five periods

Sinking Fund

Contributions to the sinking fund of a bond issue are ordinarily made at the end of the period. If a bond issue is to run twenty years and provision is to be made for it by twenty contributions to a sinking fund, the first contribution is usually made at the end of the first year, thus allowing time in which to acquire cash from profits. Such contributions to a fund constitute an ordinary annuity.

When contributions to a fund are made at the beginning of the period, the payments become an annuity due. To find the periodical contribution necessary to accumulate the required fund by such payments:

1. Find the amount of an ordinary annuity of $1 for one more than the number of periods
2. Deduct $1 in order to find the amount of an annuity due of $1 for the given number of periods
3. Divide the required fund by the amount of an annuity due of $1.

Required Annual Contribution

In the discussion of sinking fund contributions in the early part of this chapter it was required to find the sinking fund contribution necessary to invest at 4% at the end of each of five years to pay an obligation of $100,000. This was found to be $18,462.71. We shall now calculate the required annual contribution if made at the beginning of each year.

These payments constitute an annuity due of five periods. The calculation will proceed by the following steps:

1. Find the amount of an ordinary annuity of six periods:

$1.265319, amount of $1 at compound interest for six periods at 4%
— 1.

$ .265319, compound interest on $1 for six periods at 4%
$ .265319 ÷ .04 = $6.632975, amount of ordinary annuity of six periods

2. Find the amount of an annuity due of five periods:

$6.632975
— 1.

$5.632975, amount of annuity due of five periods

3. Divide total fund by amount of annuity of $1:

$100,000 ÷ 5.632975 = $17,752.61, required annual contribution

The following sinking fund table shows the accumulation of the fund due to the two elements of contributions and compound interest.

Year	Contribution	Fund Beginning Year	Interest	Fund End of Year
1	$17,752.61	$17,752.61	$ 710.10	$ 18,462.71
2	17,752.61	36,215.32	1,448.61	37,663.93
3	17,752.61	55,416.54	2,216.66	57,633.20
4	17,752.61	75,385.81	3,015.43	78,401.24
5	17,752.61	96,153.85	3,846.15	100,000.00
	$88,763.05		$11,236.95	

Present Worth of an Annuity Due

When the annuity payments are due at the beginning of the period, the present value of the annuity is composed of two elements:

1. A rent, or payment, now due (a)
2. The present value of an ordinary annuity of one less than the given number of periods (b)

For instance, $5.451822 deposited at 4% interest compounded annually permits the immediate withdrawal of:

(a) $1 rent now due, and leaves a balance of

(b) $4.451822, which is the present value of five annual payments of $1, the first of which is due one year hence.

Therefore, to find the present value of an annuity due, proceed as in the following illustration.

Illustration

What is the present value at 4% per annum of an annuity of six payments of $1, the first of which is due?

SOLUTION: 1. Find the present value of an ordinary annuity for one less than the given number of periods:

$1.000000
.821927, present value of $1 due in five years at 4%

$ .178073, compound discount on $1 due in five years at 4%
$ .178073 ÷ .04 = $4.451825, present value of an ordinary annuity of five payments

2. To this present value add one rent:

$4.451825, present value of ordinary annuity of five payments
1.00

$5.451825, present value of an annuity due of six payments

The reduction of this present value, due to the accumulation of interest and the payment of rents is, tabulated below.

Present value of annuity due of six rents	$5.451825
First rent	1.
Balance	$4.451825
Interest earned 1st period	.178073
Balance end of " "	$4.629898
Second rent	1.

Balance beginning 2d period	$3.629898
Interest earned " "	.145196
Balance end of " "	$3.775094
Third rent	1.
Balance beginning 3d period	$2.775094
Interest earned " "	.111004
Balance end of " "	$2.886098
Fourth rent	1.
Balance beginning 4th period	$1.886098
Interest earned " "	.075444
Balance end of " "	$1.961542
Fifth rent	1.
Balance beginning 5th period	$ .961542
Interest earned " "	.038461
Balance end of " "	$1.000003
Sixth rent	1.

Since the sixth rent is withdrawn at the beginning of the sixth period, there is no balance at the beginning of, nor interest earned during, the sixth period.

Rents

The converse of the above problem is to determine what periodical payment or rent a known present value will produce, when the first payment is due immediately.

For instance, what rent will an investment of $1,000 produce, the first of six annual rents being due at once and the interest being 4%?

1. Find what present value will produce a rent of $1

   This was found above to be $5.451825

2. Divide the known present values by the present value of an annuity of $1

$$\$1000.00 \div 5.451825 = \$183.42$$

The reduction of this present value may be tabulated thus:

Period	Rent	Balance	Interest	Balance
				$1,000.00
1	$183.42	$816.58	$32.66	849.24
2	183.42	665.82	26.63	692.45
3	183.42	509.03	20.36	529.39
4	183.42	345.97	13.84	359.81
5	183.42	176.39	7.06	183.45
6	183.45	0.		
	$1,100.55		$100.55	

CHAPTER XVIII

LOGARITHMS IN COMPOUND INTEREST AND ANNUITY COMPUTATIONS

Calculating Compound Interest and Annuities by Logarithms

The following examples illustrate the methods of utilizing logarithms in compound interest and annuity computations.

Examples

1. *To find the amount of 1.* Required—the amount of $4,500 at 4% per annum for twenty years, compounded semiannually

SOLUTION:

$$\text{Amount} = \$4{,}500 \times 1.02^{40}$$

Log 1.02 = .00860
Multiply by 40

Log 1.02^{40} = .34400, the log of 2.208
Hence $1.02^{40} = 2.208$
Amount = $\$4500 \times 2.208 = \$9936$
Amount computed with aid of interest table = $9936.18

Logarithms may be used more extensively in this solution, as follows:

Log 4,500 = 3.65321
Log 1.02 = .00860
Multiply by 40

Log 1.02^{40} = .34400

Log $4,500 + \log 1.02^{40} = 3.99721$, log of 9,936
Hence $\$4{,}500 \times 1.02^{40} = \$9{,}936$

2. *To find the compound interest on 1.*

SOLUTION: Compute the amount as above and deduct the principal

Amount	$9,936.
Principal	4,500.
Compound interest	$5,436.

3. *To find the principal.* What sum invested at 4% compounded quarterly will amount to $5000 in eight years?

SOLUTION:

$$\text{Principal} = 5000 \div (1.01)^{32}$$

Log 5000		= 3.69897
Log 1.01	= .00432	
Multiply by	32	
Log 1.01^{32}		= .13824
Log of principal		= 3.56073, the log of 3636.92
Principal		= $3636.92

Principal computed by interest table = $3636.52.

4. *To find the rate.* If $1,125 is to be returned at the expiration of seven years for a loan of $800, what rate of interest compounded annually is earned?

SOLUTION:

Since $1{,}125 = 800 \times (1 + i)^7$

$(1 + i)^7 = 1{,}125 \div 800$

Log 1,125 = 3.05115

Log 800 = 2.90309

Log $(1 + i)^7$ = .14806

Log $1 + i$ = .14806 ÷ 7 = .021151, the log of 1.0499

Hence $1 + i = 1.0499$

i = .0499, or nearly 5%

5. *To find the time.* For how many years should $2,000 be placed at 5% interest compounded annually to produce $5,054?

SOLUTION:

Since $\$5{,}054 = \$2{,}000 \times 1.05^n$

$5{,}054 \div 2{,}000 = 1.05^n$

Hence 3.70364, which is log 5054

Minus 3.30103, " " " 2,000

Equals .40261 " " " 1.05^n

Since .40261 = log 1.05 × n

n = .40261 ÷ log 1.05

Since log 1.05 = .02119

n = .40261 ÷ .02119 = 19, the number of years

6. *To find the present value of 1.* What is the present value of $15,000 due in five years at 4%, interest compounded quarterly?

SOLUTION:

Present value = $15,000 ÷ $(1.01)^{20}$

Log 15,000		= 4.17609
Log 1.01	= .00432	
Multiply by	20	
Log 1.01^{20}		= .08640
Log of present value		= 4.08969
Present value		= $12,293.89
Present value by compound interest table		= $12,293.17

7. *To find the compound discount on 1.*

SOLUTION: Compute the present value as in the preceding example; deduct this present value from the amount due at maturity.

Amount due at maturity	$15,000.00
Present value	12,293.89
Compound discount	$ 2,706.11

8. *To find the amount of an annuity.* What is the amount of an ordinary annuity of $25 for twenty periods at 4% per period?

SOLUTION:

Amount of an annuity of $1 = $\frac{1.04^{20} - 1}{.04}$

Log 1.04	= .01703	
Multiply by	20	
Log 1.04^{20}	= .34060, the log of 2.1908	
1.04^{20}	= 2.1908	
Deduct	1.0000	
Compound interest	= 1.1908	

Amount of annuity of $ 1 = $ 1.1908 ÷ .04 = $ 29.77

" " " " $25 = 29.77 × 25 = $744.25

" " " " " by interest tables = $744.45

Logarithms may be used more extensively in these computations, but it is desired to make the solutions as simple as possible.

9. *To find the amount of sinking fund contributions.* What annual contribution must be made at the end of each of twenty years to amount to $50,000 at 4½%?

SOLUTION:

$50,000 ÷ amount of annuity of $1 = contribution

Log 1.045 = .01912

Multiply by 20

Log 1.045^{20} = .38240, the log of 2.412111

Hence 1.045^{20} = 2.412111

Deduct 1.000000

Compound interest = 1.412111

Amount of annuity of $1 = 1.412111 ÷ .045 = 31.38

$50,000 ÷ 31.38 = $1593.37

10. *To find the present worth of an annuity.* What is the present value of an annuity of $50 per year for ten years at 6%?

SOLUTION: The present value of the annuity = $50 \times D$

$$D = 1 - \frac{1}{1.06^{10}}$$

Log 1.06 = .02531

Multiply by 10

Log of 1.06^{10} = .25310, the log of amount of 1.00

Determine the present value as follows:

Log 1. = 0.00000 or 10.00000 − 10

Log 1.06^{10} = .25310

Hence $\log \frac{1}{1.06^{10}}$ = 9.74690 − 10, the log of .5583375

1.0000000

Minus .5583375

Equals .4416625, the compound discount

.4416625 ÷ .06 = 7.36104, present value of annuity of $ 1

$7.36104 × 50 = $368.052, " " " " " $50

CHAPTER XIX

BOND DISCOUNT AND PREMIUM

Bonds Purchased below and above Par

Bonds are frequently purchased at prices either below or above par; that is, at a discount or a premium. When a bond purchased at a discount is held until maturity and paid at par, the owner makes a profit amounting to the difference between the cost and par; that is, to the discount. If the bond is purchased at a premium and repaid at par, the owner incurs a loss amounting to the premium. Conversely, one who issues a bond below par and repays it at par loses the discount, while one who issues a bond at a premium and repays it at par gains the premium. The question arises as to when such profit shall be credited, or such loss be charged, to profit and loss. The following discussion considers the subject from the point of view of the investor; the same principles apply to writing off premium or discount on the books of the business issuing the bonds.

Discount

Let us assume that a $100 bond, due in two years and bearing 4% interest payable semiannually, is bought for $96.28. When paid at par there will be a gain of $3.72, the amount of the discount. The discount may be taken up by the following methods, which are, however, unscientific:

1. By an immediate credit to income of $3.72, thus raising the investment account on the books to the par of $100 and taking credit at once for the discount. This method is clearly wrong because it takes up at the time of purchase an income which is earned gradually as the bond approaches maturity.

2. By carrying the investment at its cost of $96.28 until it is paid, at which time the difference between the cash received and the cost of the bond is credited to income. This method also is erroneous, although the error is not so apparent. In the first place, while the earning is made gradually as the bond approaches maturity, it would appear from the above treatment that none of the discount was earned until the very day of maturity; and in the second place the bond increases in value as the day of maturity approaches, when it is paid at par.

3. By considering the $3.72 as extra interest earned, dividing the amount by 4, the number of semiannual interest periods, charging ¼ of $3.72 each six months to the bond account and crediting it to interest. This has the effect of raising the investment account gradually to par, while spreading the earning over the four periods in which the bond is held. The effect of this treatment is shown by the following schedule:

Period	Bond account beginning of 6 months period	Portion of discount taken up during one period	Int. collected	Total credit to interest
1	$ 96.28	$ .93	$2.00	$ 2.93
2	97.21	.93	2.00	2.93
3	98.14	.93	2.00	2.93
4	99.07	.93	2.00	2.93
Maturity.....	100.00			
Total.....		$3.72	$8.00	$11.72

It may appear that this method answers all requirements, since it raises the investment account gradually to par, taking up the discount periodically. But it is subject to the criticism that, while the investment is gradually increasing, the interest remains constant. This means that the rate of interest is gradually decreasing, as shown in the following table:

Period	Asset value in bond account	Credit to Income	Semiannual rate of income
1	$96.28	$2.93	3.043 + %
2	97.21	2.93	3.014 + %
3	98.14	2.93	2.985 + %
4	99.07	2.93	2.957 + %

Since the investment or asset value is increasing as the bond approaches maturity, the credits to income should also increase, so that the same rate of income will be maintained each period. When the amounts are small, the error arising from this method is immaterial and the discount may be written off in equal amounts, as above. But when the amounts are large, this method may result in serious unfairness to some parties in interest. Assuming, for instance, that $96,280 is invested in the purchase of $100,000 of bonds for a trust and that the beneficiary changes at the expiration of the second six months' period, the first beneficiary would receive over 3% income semiannually on the assets of the trust, while the second beneficiary would receive less than 3%. It is not sufficient that they each receive the same number of dollars. The property, or investment, held for the second beneficiary is of greater value than that held for the first, and hence the second should receive more income per period, since each is entitled to the same rate of income on the investment of the trust.

Scientific Method of Amortization

Any method of writing off discount, to be scientific, must take the following into consideration:

1. The investment increases in value as the bond approaches maturity, and at each interest date the investment value should be increased on the books, thus raising it gradually to par.

2. The purchase of a bond at a discount results in the rate of interest earned on the investment being higher than the nominal

rate paid on the par of the bond. Hence, the amount added to the asset value of the bond at each interest date should also be credited to interest, together with the cash collected.

3. The total credit to interest (cash collected plus portion of discount) at the end of each period should be an increasing amount but always the same per cent of the carrying or asset value of the bond at the beginning of the period.

When a bond is bought at a price other than par, there are two interest rates:

1. The nominal or cash rate paid on the par of the bond
2. The effective, basic, or income rate actually earned on the investment

When the bond is bought at a discount, the income rate is greater than the nominal rate for two reasons:

1. The investment is less than par, although it gradually increases to par.
2. The income is more than the coupons collected, since the income is composed of two parts:
   (a) The coupons collected
   (b) The periodical portion of the discount

When the discount is scientifically amortized or written off the method is as follows:

1. Determine the effective rate earned on the investment, as explained later.
2. At each interest date multiply the carrying value of the investment by the effective rate, to determine the amount of revenue earned during the period.
3. Make the following entry:
   Debit cash for coupon collected
   Debit investment for portion of discount
   Credit interest for total income computed in (2) above.

The amount charged to investment will be the difference between the total income and the cash collected.

To illustrate, the price of $96.28 in the example above, was chosen because it was known that a 4% bond due in two years, with semiannual interest, would yield an effective rate of income of 6% per year or 3% semiannually if purchased at this price. The following shows the scientific amortization of the discount:

Cost of bond		$96.28
First period:		
Income, 3% of $96.28	$2.89	
Coupon, 2% of $100.00	2.00	
Balance—portion of discount charged to investment		.89
Second period:		
Carrying value of bond		$97.17
Income, 3% of $97.17	$2.92	
Coupon	2.00	
Balance—portion of discount charged to investment		.92
Third period:		
Carrying value of bond		$98.09
Income, 3% of $98.09	$2.94	
Coupon	2.00	
Balance—portion of discount charged to investment		.94
Fourth period:		
Carrying value of bond		$99.03
Income, 3% of $99.03	$2.97	
Coupon	2.00	
Balance—portion of discount charged to investment		.97
Par of bond		$100.00

It will be noted that this process conforms to the three requirements of the scientific method as outlined above.

The investment is increased gradually to par by writing off a portion of the discount at each interest period, thus:

Beginning of Period	Investment
1	$96.28
2	97.17
3	98.09
4	99.03
Par	100.00

The amount added to the investment value at each interest period is also credited to interest, as follows:

Credits to Interest

End of Period	For Cash	For Portion of Discount	Total
1	$2.00	$ .89	$2.89
2	2.00	.92	2.92
3	2.00	.94	2.94
4	2.00	.97	2.97

The total credit to interest increases each period and is always the same per cent of the increased investment at the beginning of each period, as shown below:

Period	Investment, beginning of period	Income for period	Rate
1	$96.28	$2.89	3%
2	97.17	2.92	3%
3	98.09	2.94	3%
4	99.03	2.97	3%

While, as stated before, the difference is immaterial when the investment is small, the following table comparing results by the equal instalment method and the scientific amortization method will serve to show the injustice which might be caused by the improper method.

Period	Investment Value			Income		
	Equal instal. plan	Scientific amortzn. plan	Difference	Equal instal. plan	Scientific amortzn. plan	Difference
1	$96.28	$96.28	$.00	$2.93	$2.89	$+.04
2	97.21	97.17	.04	2.93	2.92	+.01
3	98.14	98.09	.05	2.93	2.94	−.01
4	99.07	99.03	.04	2.93	2.97	−.04

Assuming that the bond was held as an investment for a trust and that the beneficiary changed at the end of the second period, the column of income differences shows that, by the equal instalment method, the first beneficiary would receive 5 cents too much income, while the second beneficiary would receive 5 cents less than was rightfully his.

Assuming that the bond was held for an estate for the benefit of a life tenant and a remainderman, that the life tenant died at the end of the second half year, and that the estate reverting to the remainderman, the equal instalment method would work detriment to the remainderman. Let us assume that the estate originally consisted of $100. By the equal instalment method the estate reverting to the remainderman would be less than $100.

The income represented by the portion of discount written off would be paid to the life tenant in cash and would be added to the carrying value of the bond.

The following would be a statement of the assets of the estate:

	Equal Instalment Method			Scientific Amortization Method		
	Cash	Bond true value	Total	Cash	Bond true value	Total
Original	$100.00		$100.00	$100.00		
Bond purchase	96.28			96.28		
	$ 3.72	$96.28	100.00	$ 3.72	$96.28	$100.00
First period:						
Cash collected	2.00			2.00		
	$ 5.72			$ 5.72		
Income paid L. T.	2.93			2.89		
Assets of estate	$ 2.79	97.17	99.96	$ 2.83	97.17	100.00
Second period:						
Cash collected	2.00			2.00		
	$ 4.79			$ 4.83		
Income paid L. T.	2.93			2.92		
Assets of estate	$ 1.86	98.09	99.95	$ 1.91	98.09	100.00

It is seen that if the equal instalment method was followed, the assets of the estate would be impaired 5 cents by paying the life tenant in cash 5 cents erroneously computed as income. The result would be that the remainderman would be defrauded, the injustice being concealed by turning the bond back to him at a value of $98.14 when it really had a value of only $98.09.

Income Rates

Bond dealers frequently offer bonds at prices which net the investor a rate other than the cash or nominal rate. In the illustration above, where a 4% bond was purchased to yield an income rate of 6%, the quotation might be stated, "4% bond paying 6%," "4% bond yielding 6%," "4% bond to net 6%," or "4% bond on a 6% basis." To avoid the necessity of computing the price at which each sale must be made, bond tables have

been prepared showing the price to be paid. This price varies with

1. The nominal rate
2. The income rate
3. The number of periods until maturity

Since bonds usually bear semiannual interest there are two periods per year, and bond tables usually show a price based on the assumption that the interest is paid semiannually.

The following portion of a page in a bond table shows various prices to be paid for a 2-year bond, depending on the nominal and effective rates. The rates at the head of the columns are nominal rates; those at the side are the effective rates. For instance, in the 4% nominal rate column on the 6% effective rate line is found the price of $96.28 used in the preceding illustrations of a 2-year 4% bond netting 6%.

	2 Years Interest Payable Semiannually						
Per Cent Per Annum	3%	3 1/2%	4%	4 1/2%	5%	6%	7%
4.80	96.61	97.55	98.49	99.43	100.38	102.26	104.15
4 7/8	96.47	97.41	98.35	99.29	100.24	102.12	104.00
4.90	96.42	97.36	98.31	99.25	100.19	102.07	103.95
5	96.24	97.18	98.12	99.06	100.00	101.88	103.76
5.10	96.05	96.99	97.93	98.87	99.81	101.69	103.57
5 1/8	96.01	96.95	97.89	98.83	99.77	101.64	103.52
5.20	95.87	96.81	97.75	98.69	99.62	101.50	103.38
5 1/4	95.78	96.72	97.66	98.59	99.53	101.41	103.28
5.30	95.69	96.63	97.56	98.50	99.44	101.31	103.19
5 3/8	95.55	96.49	97.43	98.36	99.30	101.17	103.04
5.40	95.51	96.44	97.38	98.32	99.25	101.12	103.00
5 1/2	95.33	96.26	97.20	98.13	99.07	100.93	102.80
5 5/8	95.10	96.03	96.97	97.90	98.83	100.70	102.57
5 3/4	94.87	95.81	96.74	97.67	98.60	100.47	102.33
5 7/8	94.65	95.58	96.51	97.44	98.37	100.23	102.09
6	94.42	95.35	96.28	97.21	98.14	100.00	101.86

Bond Premium

When a bond is bought at a price above par, the effective rate is less than the nominal rate, for two reasons:

1. The investment is more than the par to which the nominal rate applies.
2. The coupons collected are not all income; since only par will be repaid, a portion of the cash received at each interest date must be considered as a return of the premium (which is principal) and only the balance as income.

Bond premium should be scientifically written off as follows:

1. Determine the effective rate.
2. Multiply the gradually diminishing investment by the effective rate to compute the income earned during the period.
3. Make the following entry:
   Debit cash for coupon collected
   Credit income for amount computed in (2) above
   " investment for difference between cash and income.

To illustrate, the bond table shows that a 2-year $100 6% bond to net 4% should be purchased for $103.81. The following shows the scientific amortization of the premium:

Cost of bond		$103.81
First period:		
Coupon, 3% of $100.00	$3.00	
Income, 2% of $103.81	2.08	
Balance, premium written off		.92
Second period:		
Carrying value of bond		$102.89
Coupon	$3.00	
Income, 2% of $102.89	2.06	
Balance, premium written off		.94

Third period:		
Carrying value of bond		$101.95
Coupon	$3.00	
Income, 2% of $101.95	2.04	
Balance, premium written off		.96
Fourth period:		
Carrying value of bond		$100.99
Coupon	$3.00	
Income, 2% of $100.99	2.02	
Balance, premium written off		.98
Par of bond		$100.01

The error of 1 cent arises from repeated approximations of interest to the nearest cent.

The following table shows the periodical entries:

Period	Debit Cash	Credit Interest	Credit Investment (or Premium)	Investment
Cost				$103.81
1	$ 3.00	$2.08	$ .92	102.89
2	3.00	2.06	.94	101.95
3	3.00	2.04	.96	101.99
4	3.00	2.01*	.99*	100.00
Total	$12.00	$8.19	$3.81	

* Adjustment of 1 cent to correct discrepancy.

The second column shows the total cash collected; the third, the amount of cash taken as income; the fourth, the portion of the coupon applied to repayment of the premium ($3.81) which, with the final repayment of the bond, completely realizes the investment. Since a portion of the investment is realized periodically, the investment gradually diminishes in value. Hence the credits to interest decrease with each successive period.

Computing the Premium and the Price

While the price above par to be paid for a bond to yield an effective rate less than the nominal rate may be found in a bond table, such tables are not always available. If a table of compound interest, present values or annuities is at hand, it will be of assistance, but the price may be computed without tables of any kind. Two methods are explained, as follows:

First Method

An interest bearing bond comprises two promises, as follows:

1. To pay the par at maturity—in the last preceding illustration, $100 at the expiration of four six-months periods.
2. To pay a stipulated amount of interest periodically—in the illustration, $3 at the end of each of four six-months periods.

The price to be paid for the bond is the present value, discounted at the effective rate, of all cash payments promised. In the illustration, the price is the sum of

1. The present value of $100 due four periods hence, discounted at 2% per period.
2. The present value of an annuity of $3 for four periods, discounted at 2% per period.

These present values may be found in the interest table; thus the table shows that:

Present value of $1 due four periods hence at 2% = $.923845

$.923845 × 100 = $92.3845, P. V. of principal

Present value of an annuity of $1 for four periods at 2% = $3.807729

$3.807729 × 3 = 11.4232, P. V. of coupons

Total	$103.8077
or	$103.81, price

If the table does not show present values but does show compound interest, the computation may be made as follows:

The table shows the amount of $1 at compound interest for four periods at 2% to be $1.082432.

Then, $1.00 ÷ 1.082432 = $ .923845, P. V. of $1	
$.923845 × 100	= $ 92.3845, P. V. of principal
Also, $1.00 − .923845 = $ .076155,	
compound discount, four periods	
$.076155 ÷ .02 = $ 3.8077,	
P. V. of annuity of $1 for four periods	
$3.8077 × 3	= 11.4231, P. V. of coupons
Total	$103.8076, price

When no interest table of any kind is available, recourse may be had to the methods explained in the chapter on compound interest and annuities. Perhaps the easiest method would be to compute the amount of $1 for four periods at 2% thus:

1.02
1.02
1.0404
1.0404
1.082432

The procedure following would be as shown above.

Second Method

This method determines only the premium to be paid to reduce the income from the nominal rate to the effective rate. The premium so determined, added to the par of the bond, comprises the total price. The method is based on the following reasoning:

Let us assume that a 4% annual, or 2% semiannual, income is required. If the bond bore 4% interest, the price would be par. Hence the payment of $100, or par, entitles the holder of the bond to receive:

1. Principal, at maturity
2. Interest, $2 at the end of each six months

But if, as in the preceding illustration, the bond pays 3% semiannually, each semiannual coupon collected will be $3. Of this, the payment of par entitles the holder to receive $2. And the payment of a premium entitles him to receive the remaining $1. Hence, the premium is the sum which must be paid to entitle the holder to collect that portion of the periodical coupon which is in excess of the product obtained by multiplying the par of the bond by the effective periodical rate.

This excess interest is an annuity. In the case of a 2-year 6% bond bought to net 4%, the annuity is $1 for four periods. The premium is the present value of this annuity, discounted at the effective rate, and may be computed as follows:

$100 × 3% (semiannual cash rate)	= $3.00 coupon
$100 × 2% (" effective rate)	= 2.00 effective income on par
Excess	$1.00

The present value at 2% of an annuity of $1 for four periods is shown by an interest table to be $3.807729, to which is added the par, $100, the total being the price of the bond or $103.81.

When a table showing the present value of an annuity is not at hand, one of the methods already explained may be used to find the present value. Perhaps the easiest method is by successive divisions, as follows:

Example

What is the premium to be paid on a 3-year 7% bond, interest payable semiannually, bought to net 5%; par $100,000?

SOLUTION:

$100,000 × 3½% (cash rate per period)	= $3,500
$100,000 × 2½% (effective rate per period)	= 2,500
Excess	$1,000

The required premium, therefore, is the present value at 2½% of an annuity of $1,000 for six periods.

Now,

$1,000.00 ÷ 1.025 =	$ 975.610	P. V. of $1000 due 1 period hence	
975.610 ÷ 1.025 =	951.814	" " " $1000 " 2 periods	"
951.814 ÷ 1.025 =	928.599	" " " $1000 " 3 "	"
928.599 ÷ 1.025 =	905.950	" " " $1000 " 4 "	"
905.950 ÷ 1.025 =	883.854	" " " $1000 " 5 "	"
883.854 ÷ 1.025 =	862.297	" " " $1000 " 6 "	"

$ 5,508.124, premium
100,000.000, par

$105,508.12, price

The premium could also be computed as follows: Find amount of $1 for six periods at 2½%, as follows:

1.025
1.025

1.050625 amount of $1 for 2 periods
1.050625

1.103813 " " 1 " 4 "
1.050625

1.159693 " " 1 " 6 "

Find compound discount on $1 due six periods hence at 2½ % thus:

$1 ÷ 1.159693 = $.862297 P. V. of $1 due six periods hence
$1 − $ .862297 = .137703 compound discount
or $.159693 (comp. int.) ÷ 1.159693 (amt.) = $.137703 compound discount

Find present value of annuity of $1,000 for six periods at 2½ %, thus:

$ .137703 ÷ .025 = $5.50812 P. V. of annuity of $1
$5.50812 × 1000 = $5,508.12 P. V. of annuity of $1,000

The following table shows the periodical entries and the diminishing balance of the investment.

Period	Debit Cash	Credit Interest 2½% of Investment	Credit Investment	Investment
Cost				$105,508.13
1	$ 3,500	$ 2,637.70	$ 862.30	104,645.83
2	3,500	2,616.15	883.85	103,761.98
3	3,500	2,594.05	905.95	102,856.03
4	3,500	2,571.40	928.60	101,927.43
5	3,500	2,548.19	951.81	100,975.62
6	3,500	2,524.38	975.62	100,000.00
Total	$21,000	$15,491.87	$5,508.13	

Computing the Discount and the Price

When bond tables cannot be consulted to determine the price to pay for a bond to net an income rate higher than the nominal rate, the price may be computed by methods similar to those described for determining a premium.

First Method. The first method consists of the following three steps:

(a) Compute the present value of the par, discounted at the effective rate.

(b) Compute the present value at the effective rate of all coupons to be collected.

(c) Add the foregoing two items, the sum being the price to be paid.

In the case of the $100 2-year 4% bond bought to net 6%, at a price of $96.28, the first item (a) is the present value of $100 due four periods hence at 3% (the effective rate) per period. The second item (b) is the present value of an annuity of $2 for four periods discounted at 3%.

These present values may be found in a book of tables, thus:

Present value of par:		
P. V. of $1 due in four periods at 3% is	$ .888487	
$.888487 × 100 =		$88.8487
Present value of coupons:		
P. V. of annuity of $1 for four periods at 3% is $3.717098		
$3.717098 × 2 =		7.4342
Total price		$96.2829

or as shown by the bond table $96.28

If an interest table is not available, the methods already described for determining present values may be used.

2. *Second Method.* This method determines the discount to be deducted from par. Since the cash rate does not produce the required income, the seller permits the deduction of an amount which, invested at the effective rate, will produce the extra periodical income required. The discount is the present value of an annuity of the extra income.

For instance, in the case of the $100 2-year 4% bond bought to net 6%, at a price of $96.28, the 3% effective rate is equivalent to $3 per period, while the 2% coupon produces only $2 per period, the required excess being $1 per period for four periods. The present value of 3% (the effective rate) of an annuity of $1 for four periods is shown by an interest table to be $3.717098.

Then	$100.00 par
Less	3.72 discount
	$96.28 price

When neither a bond table nor an interest table can be used, one must resort to the previously described method of computing the present value of the annuity.

Illustration

What are the discount and the purchase price of a 3-year 5% bond, interest payable semiannually, bought to net 6%; par $100,000?

SOLUTION:

$100,000 × 3% (effective rate per period)	$3,000
$100,000 × 2½% (cash " " ")	2,500
Deficient interest per period	$ 500

The required discount, therefore, is the present value at 3% of an annuity of $500 per period.

Now,	$500	÷ 1.03 =	$485.437	P. V. of	$500	due 1	period
Hence,	485.437	÷ 1.03 =	471.298	" "	500	" 2	periods
"	471.298	÷ 1.03 =	457.571	" "	500	" 3	"
"	457.571	÷ 1.03 =	444.244	" "	500	" 4	"
"	444.244	÷ 1.03 =	431.305	" "	500	" 5	"
"	431.305	÷ 1.03 =	418.742	" "	500	" 6	"
Discount			$2,708.597				

Then par	$100,000.00
Less discount	2,708.60
Price	$ 97,291.40

The discount can also be computed as follows:

Find the amount of $1 for six periods at 3%, thus:

1.03	
1.03	
1.0609	amount of $1 for 2 periods
1.0609	
1.125509	" " 1 " 4 "
1.0609	
1.194052	" " 1 " 6 "

Find the compound discount on $1 due six periods hence at 3%, thus:

$1.00 ÷ 1.194052 = $.837484 P. V. of $1 due six periods hence
$1.00 − .837484 = $.162516 compound discount

Find the present value of an annuity of $500 for six periods at 3% thus:

$.162516 (comp. dis.) ÷ .03 = $5.4172 P. V. of annuity of $1
$5.4172 × 500 = $2708.60 present value of annuity of $500, or discount

The following table, or schedule of amortization, shows the periodical entries and the remaining balance of the investment.

Period	Credit Income 3% of Investment	Debit Cash	Debit Investment	Investment
				$ 97,291.40
1	$ 2,918.74	$ 2,500	$ 418.74	97,710.14
2	2,931.31	2,500	431.31	98,141.45
3	2,944.24	2,500	444.24	98,585.69
4	2,957.57	2,500	457.57	99,043.26
5	2,971.30	2,500	471.30	99,514.56
6	2,985.44	2,500	485.44	100,000.00
	$17,708.60	$15,000.00	$2,708.60	

Purchases at Intermediate Date

In the preceding explanations of methods for computing prices for bonds either at a premium or at a discount, it has been assumed that the purchase occurred on an interest date. When this is not the case, the customary method of determining the price is as follows:

1. Compute the price as if the purchase had been made at the next preceding interest date; also the price as if the purchase were to be made at the next succeeding interest date. The difference is the portion of premium or discount to be amortized during the period.

2. Such a proportion of this premium or discount is amortized as the elapsed time between the preceding interest date and the date of purchase bears to the total interest period.

3. To the amortized value thus obtained add the accrued interest at the nominal rate.

For instance, referring to the table on page 209 let us assume that the interest dates are January 1 and July 1; and that

$105,508.13 is the value on Jan. 1, 1918, on a 5% basis
104,645.83 " " " " July 1, 1918, " " " "
$ 862.30 " " premium to be amortized during the period

If the purchase is made on February 1, 1918, one-sixth of the interest period has elapsed, hence one-sixth of $862.30 should be amortized, thus:

Value at January 1, 1918	$105,508.13
Deduct $^1/_6$ of $862.30	143.72
	$105,364.41
Add accrued interest: $^1/_6$ of $3500	583.33
Price	$105,947.74

The price may also be computed by adding to the price on January 1 the accrued interest at the effective rate, thus:

Price on January 1, 1918	$105,508.13
Effective interest for six months:	
2½% of $105,508.13 = $2,637.70	
$^1/_6$ of $2,637.70	439.61
Price on February 1, 1918	$105,947.74

Of this amount $105,364.41 is charged to investment and $583.33 to accrued interest. When the coupon is collected on July 1, it is applied as follows:

Cash	$3500.00	
Accrued interest		$ 583.33
Interest earned		2,198.09
Investment amortization ($862.30 − $143.72)		718.58

The investment of $105,364.41, reduced by the amortization of $718.58, is now carried at the true value (see page 209) on July 1, $104,645.83.

While this is the customary method of computing a flat price, it is unfair to the buyer, since he advances to the seller $583.33 accrued interest five months before it is due.

When the bond is to be purchased at a discount, the proportion of discount to be amortized for the fractional period should be added. Referring to page 212, let us assume that interest is payable on January 1 and July 1, that $97,291.40 is the value at

January 1, 1918, on a 6% basis, and that the transfer is to be made on May 1, 1918.

Then, $97,710.14 is the value on July 1, 1918
97,291.40 " " " " January 1, 1918
$ 418.74 " " discount to be amortized during the period
⅔ of $418.74 = $ 279.16 the discount to be amortized during two-thirds of the period

$97,291.40 value at Jan. 1, 1918
279.16 two-thirds of discount amortized in six months
1,666.67 accrued interest—two-thirds of $2,500
$99,237.23 price on May 1, 1918

Or, value at January 1, 1918	$97,291.40
Add two-thirds of 3% (effective rate) on $97,291.40	1,945.83
	$99,237.23

Of this amount $97,570.56 is charged to investment and $1,666.67 to accrued interest. On July 1, the investment is adjusted to its true value (see table on page 212) by the following entries:

Cash	$2500.00	
Investment (discount $418.74 − $279.16)	139.58	
Accrued interest		$1666.67
Interest		972.91

All entries after July 1, 1918, are as indicated in the schedule on page 212.

Serial Bonds

Instead of providing a sinking fund for the eventual redemption of a bond issue, the bonds may be retired gradually by serial redemption. In computing the price at which the entire issue may be purchased to net an effective rate other than the cash rate, the bonds maturing at each redemption date must be considered separately, the several values so obtained being added to find the total price.

As a simple illustration assume that five bonds of $100 each, bearing 6% interest payable semiannually, are to be retired in amounts of $100 at the end of each of five years. Required—the price to net 5%. The following values are taken from a bond table, though they could be computed by the methods already explained.

Maturities	Value
Bond due in 1 year	$100.96
" " " 2 years	101.88
" " " 3 "	102.75
" " " 4 "	103.59
" " " 5 "	104.38
Total value of issue	$513.56

The following schedule shows the reduction of the premium and the serial redemption of the bonds:

Cost		$513.56
First period:		
Coupons 3% of $500	$15.00	
Income 2 1/2% of $513.56	12.84	2.16
Second period:		
Carrying value		$511.40
Coupons 3% of $500	$15.00	
Income 2 1/2% of $511.40	12.78	2.22
		$509.18
First redemption		100.00
Third period:		
Carrying value		$409.18
Coupons 3% of $400	$12.00	
Income 2 1/2% of $409.18	10.23	1.77
Fourth period:		
Carrying value		$407.41
Coupons 3% of $400	$12.00	
Income 2 1/2% of $407.41	10.18	1.82
		$405.59

Second redemption		100.00
Fifth period:		
Carrying value		$305.59
Coupons 3% of $300	$ 9.00	
Income 2 1/2% of $305.59	7.64	1.36
Sixth period:		
Carrying value		$304.23
Coupons 3% of $300	$ 9.00	
Income 2 1/2% of $304.23	7.61	1.39
		$302.84
Third redemption		100.00
Seventh period:		
Carrying value		$202.84
Coupons 3% of $200	$ 6.00	
Income 2 1/2% of $202.84	5.07	.93
Eighth period:		
Carrying value		$201.91
Coupons 3% of $200	$ 6.00	
Income 2 1/2% of $201.91	5.05	.95
		$200.96
Fourth redemption		100.00
Ninth period:		
Carrying value		$100.96
Coupons 3% of $100	$ 3.00	
Income 2 1/2% of $100.96	2.53	.47
Tenth period:		
Carrying value		$100.49
Coupons 3% of $100	$ 3.00	
Income 2 1/2% of $100.49	2.51	.49
		$100.00
Fifth redemption		100.00
		0

CHAPTER XX

LEASEHOLDS

Commuted Rents

The problem of determining the present value of an annuity arises when real estate is leased and an advance payment is made covering, or applying on, a series of rents which would otherwise be paid at regular intervals in the future.

Let us assume that it is proposed to lease certain property on January 1, 1918, for five years at an annual rental of $1,000, payable on January 1 of each year. This contract would require the following payments:

Date	Rent
January 1, 1918	$1,000
" 1, 1919	1,000
" 1, 1920	1,000
" 1, 1921	1,000
" 1, 1922	1,000

The lessee desires to make one payment on January 1, 1918, covering the entire rental; and it is agreed between the parties to discount at 5% the payments which would otherwise be made in 1919, 1920, 1921 and 1922. The single payment to be made (i.e., the present value of the annuity) may be computed thus:

Date	Due	Rent	Symbols	Present Value
January 1, 1918	Now	$1,000	1	$1,000.000
" 1, 1919	1 year hence	1,000	$1 \div (1 + i)$	952.381
" 1, 1920	2 years "	1,000	$1 \div (1 + i)^2$	907.029
" 1, 1921	3 " "	1,000	$1 \div (1 + i)^3$	863.838
" 1, 1922	4 " "	1,000	$1 \div (1 + i)^4$	822.702
Single payment				$4,545.950

The single payment can also be computed thus:

Present value of first payment		$1,000.00
Present value of future payments:		
Present value of an annuity of $1,000 for four periods at 5%		
Present value of $1 due in four years at 5%	$.822702	
Compound discount	$.177298	
Present value annuity of $1 = $.177298 ÷ .05	$3.54596	
$3.54596 × 1,000		3,545.96
Single payment		$4,545.96

Although the single payment of $4,545.96 pays the rent for five years, the annual entries in the accounts must show the following:

1. Annual rental of $1,000
2. Annual interest earning on the advance payment
3. Application of advance payment to annual rent
4. Reduction of value of advance payment

The following schedule shows the figures used in the annual journal entries.

Date	Debit Rent	Credit Interest	Credit Leasehold	Debit balance of Leasehold Account
Cost of leasehold				$4,545.95
January 1, 1918	$1,000		$1,000.00	3,545.95
" 1, 1919	1,000	$177.30	822.70	2,723.25
" 1, 1920	1,000	136.16	863.84	1,859.41
" 1, 1921	1,000	92.97	907.03	952.38
" 1, 1922	1,000	47.62	952.38	0
Totals	$5,000	$454.05	$4,545.95	

It is manifest that the value of a leasehold depends largely on the rate of interest used. If the above lease had been com-

muted on a basis of 4% its value would have been $4,629.90, or $83.95 more than it is worth at 5%. On a very short lease the difference is not very great, but on a 99-year lease it amounts to a very large sum.

Sublease

A similar problem arises when property is subleased. Let us assume that A leases certain property on January 1, 1912, for a period of ten years at an annual rental of $3,000, and that he occupies the property until December 31, 1917, at which time he assigns the lease to B. Due to a rise in land values, B is willing to assume the lease at an annual rental of $5,000. B may, therefore, make the following payments:

Date	Payment to Owner of Fee	Payment to A
January 1, 1918	$3,000	$2,000
" 1, 1919	3,000	2,000
" 1, 1920	3,000	2,000
" 1, 1921	3,000	2,000
" 1, 1922	3,000	2,000

Or B may, with A's consent, pay A a lump sum on January 1, 1918, instead of annual payments. If the future payments of this annuity of $2,000 are discounted at 5%, the lump sum is computed as follows:

Present value of first payment	$2,000.00
Present value of an annuity of $2,000 for four periods at 5%:	
Present value of an annuity of $1,000 (as computed above) $3,545.95	
Then $3,545.95 × 2 =	7,091.90
Single payment	$9,091.90

The following schedule shows the journal and cash entries to be made annually by B.

Date	Debit Rent	Credit Interest	Credit Leasehold	Credit Cash (paid owner)	Balance of Leasehold Account
Cost of leasehold . . .					$9,091.90
January 1, 1918	$ 5,000		$2,000.00	$ 3,000.00	7,091.90
" 1, 1919	5,000	$354.60	1,645.40	3,000.00	5,446.50
" 1, 1920	5,000	272.33	1,727.67	3,000.00	3,718.83
" 1, 1921	5,000	185.94	1,814.06	3,000.00	1,904.77
" 1, 1922	5,000	95.24	1,904.76	3,000.00	.01
Total	$25,000	$908.11	$9,091.89	$15,000.00	

CHAPTER XXI

DEPRECIATION METHODS

Annual Depreciation

There are six methods in more or less general use for the determination of the amount of depreciation which business enterprises should provide annually on their fixed assets. Writing off an arbitrary amount at any convenient time is not included, because it is too haphazard and unscientific to be called a method. The six recognized methods are as follows:

1. Straight line method
2. Diminishing value, using a fixed per cent
3. Diminishing value method based on the sum of the years' digits
4. Annuity method
5. Sinking fund method
6. Production method

It is not within the province of this book to examine the merits of the different methods. The object is only to indicate how the amount to be written off annually is ascertained when the managers of the business have decided upon the method to be used.

The illustration to be used throughout is that of an asset costing $5,000 which is estimated to have a life of six years and a residual or scrap value of $200, if any.

1. The straight line method has the advantage of simplicity, as the annual amount is determined by the simple process of dividing the total depreciation by the number of years the asset is expected to remain effective, as shown below.

Year		Annual Depreciation	Carrying Value
			$5,000
1		$ 800	4,200
2		800	3,400
3		800	2,600
4		800	1,800
5		800	1,000
6		800	200
Total		$4,800	

2. The diminishing value method, when a fixed per cent is applied, is a desirable one to use, but it has the disadvantage that the rate on the diminishing value is difficult to compute. The process requires the extraction of a root the index of which is the same as the number of years covered by the depreciation.

The formula for this method is

$$r = 1 - \sqrt[n]{\frac{s}{c}}$$

In this formula r represents the desired rate; n the number of periods; s the scrap value; and c the cost. The formula is applied to the illustration as follows:

$$r = 1 - \sqrt[6]{\frac{200}{5,000}}$$

$$= 1 - \sqrt[6]{.04}$$

Log .04 is..... 8.60206 − 10
Add..... 50. − 50

58.60206 − 60
Divide by 6..... 9.76701 − 10 log of .5848

$$r = 1 - .5848$$
$$= .4152$$
$$= 41.52\%$$

The following table shows the depreciation charges and the diminishing value in the example taken.

Year		Depreciation	Diminishing value
			$5,000.00
1	41.52% of $5,000.00	$2,076.00	2,924.00
2	" " 2,924.00	1,214.04	1,709.96
3	" " 1,709.96	709.98	999.98
4	" " 999.98	415.19	584.79
5	" " 584.79	242.80	341.99
6	" " 341.99	141.99	200.00
Total.....		$4,800.00	

The fixed per cent of diminishing value is very frequently used, but because of the difficulty or ignorance of the method of computing it, the rate is guessed at, or a purely arbitrary one is used, which, however, is usually too small. Thus, it might seem to many factory managers that 25% would be too large a rate, in the preceding illustration but it really is not, as the following shows:

Year	25% of Diminishing value	Diminishing value
		$5,000.00
1	$1,250.00	3,750.00
2	937.50	2,812.50
3	703.13	2,109.37
4	527.34	1,582.03
5	395.51	1,186.52
6	296.63	889.89
Total.....	$4,110.11	

3. The diminishing value method based on the sum of the years' digits is one which is often used to avoid the difficulties of the percentage method described above. To find the sum of the years' digits the figures representing the successive years are added together to form a denominator, in our illustration, 21

(1+2+3+4+5+6). Each year a numerator is used representing the successive number of years the asset is expected to live, and the fraction thus obtained is applied to the estimated total depreciation to determine the amount to be charged off that year. The result of applying this method in our illustration is as follows:

Year	Fraction	Annual Charge	Carrying Value
			$5,000.00
1	6/21	$1,371.43	3,628.57
2	5/21	1,142.86	2,485.71
3	4/21	914.29	1,571.42
4	3/21	685.71	885.71
5	2/21	457.14	428.57
6	1/21	228.57	200.00
Total...	21/21	$4,800.00	

As is readily seen, this method spreads the depreciation with less difference between the early and the late years than does the second method.

4. The annuity method, to quote Hatfield

> . . . rests upon the assumption that the cost of production includes not only repairs and the depreciation of machinery, but as well interest on the amount of capital invested in the machine. Depreciation, on this theory, should be a sum figured as a constant annual charge, sufficient not only to write off the decline in value but also to write off annual interest charges on its diminishing value.[1]

In other words, this method treats the cost of machinery as an investment earning interest. Hence the cost of machinery is dealt with as the present value of an annuity; the depreciation to be written off periodically is an equal amount, and the credit to interest decreases periodically because of the diminishing value of the asset.

[1] H. R. Hatfield, Modern Accounting, 1911, p. 131.

If there were no scrap value, the computation of the annual depreciation charge would be as follows: The cost of the machine is the present value of an annuity of unknown rents, or depreciation charges; this cost is divided by the present value of an annuity with rents of $1. If it is assumed that the asset in our illustration will have no scrap value and that the annuity is to be based on an interest rate of 6%, the computation is as follows:

Present value of $1 due six periods hence....	$ .70406054
Compound discount ($1 — $.70496054).....	.29503946
Present value of annuity of $1 ($.29503946 ÷ .06).....	4.917324
Annual amount required ($5,000 ÷ 4.917324).	1,016.81

The following is a table of depreciation for our example, based on the annuity method, assuming the asset has no scrap value.

Year	Debit Depreciation	Credit Interest	Credit Depreciation Reserve	Balance-carrying value Basis of interest
				$5,000.00
1	$1,016.81	$ 300.00	$ 716.81	4,283.19
2	1,016.81	256.99	759.82	3,523.37
3	1,016.81	211.40	805.41	2,717.96
4	1,016.81	163.08	853.73	1,864.23
5	1,016.81	111.85	904.96	959.27
6	1,016.81	57.56	959.25	.02
Total.....	$6,100.86	$1,100.88	$4,999.98	

When the asset has a scrap value, the conditions are more complicated, because the cost of the asset consists of two elements:

(a) The present worth of an annuity of the depreciation charges, and
(b) The present worth of the scrap value.

Referring to our illustration, if the asset has a scrap value of $200, the $5,000 invested in it is the sum of the following two items:

(a) The present value at 6% of $200 remaining after six years

(b) The present value on the basis of 6% of an annuity of six rents of unknown amounts

The rents, or depreciation charges, are computed thus:

Cost of asset	$5,000.00
Deduct present value of $200 due in six years (.70496054 × 200)	140.99
Present value of six depreciation charges	$4,859.01

In the preceding illustration the present value of an annuity of $1 for six periods at 6% was found to be $4.917324.

Then $4,859.01 ÷ 4.917324 = $988.14 annual depreciation

The following is the table of depreciation based on the annuity method, when the asset has a scrap value of $200.

Year	Debit Depreciation	Credit Interest	Credit Depreciation Reserve	Balance
				$5,000.00
1	$ 988.14	$ 300.00	$ 688.14	4,311.86
2	988.14	258.71	729.43	3,582.43
3	988.14	214.95	773.19	2,809.24
4	988.14	168.55	819.59	1,989.65
5	988.14	119.38	868.76	1,120.89
6	988.14	67.25	920.89	200.00
Total	$5,928.84	$1,128.84	$4,800.00	

It is not pertinent here to enter into a discussion of the propriety of including interest on fixed assets among the manufacturing expenses.

5. The sinking fund method is based on the assumption that a fund is set aside to accumulate at compound interest with which to acquire a new asset when the old one is discarded. It assumes that the funds for the purchase of the asset will be provided from two sources:

(a) The scrap value of the old asset
(b) The sinking fund

Since the fund accumulates at compound interest, it is a sinking fund in the mathematical sense, but not in the accounting sense, which limits the term "sinking fund" to a fund accumulated to pay a definite liability. It would be preferable to call this fund a replacement fund accumulated on the sinking fund principle.

At the end of the life of the asset the fund should equal the amount of the depreciation. The annual contributions to the fund will be the rents of an annuity which will produce this total depreciation fund.

In our illustration the total depreciation is $4,800, or cost minus scrap value $(c-s)$. Hence the annual contribution to the fund (SFC) will be $4,800 divided by the amount of an annuity of $1 for the given time and at the given rate. This is found by dividing the compound interest on $1 for the given time by the given rate of interest $(I \div i)$. The formula is:

$$SFC = (c - s) \div \frac{I}{i}$$

or

$$SFC = (c - s) \times \frac{i}{I}$$

Assuming that a fund is to be accumulated on a 4% basis, the compound interest (I) on $1 at 4% for six years being $0.265319, the formula is applied thus:

$$SFC = (\$5{,}000 - \$200) \times \frac{.04}{.265319}$$

$$= \$4{,}800 \times \frac{.04}{.265319}$$

$$= \frac{\$192}{.265319}$$

$$= \$723.66$$

If a fund is established, the entries therefor will be a debit to fund and a credit to cash each year for $723.66. When the interest is collected, the cash goes into the fund by an entry debiting fund and crediting interest. In addition there must be entries for depreciation, debiting depreciation and crediting reserve for depreciation with an amount equal to the sum of the cash contributed and the interest earned each year.

The following table shows the operation of the sinking fund method:

End of Year	Credit Cash	Credit Interest	Debit Fund	Total Fund and Reserve	Carrying Value
					$5,000.00
1	$ 723.66		$ 723.66	$ 723.66	4,276.34
2	723.66	$ 28.95	752.61	1,476.27	3,523.73
3	723.66	59.05	782.71	2,258.98	2,741.02
4	723.66	90.36	814.02	3,073.00	1,927.00
5	723.66	122.92	846.58	3,919.58	1,080.42
6	723.66	156.78	880.44	4,800.02	199.98
Total....	$4,341.96	$458.06	$4,800.02		

The annual charges to depreciation are the amounts in the column headed "Debit Fund." Thus the reserve for depreciation is always equal to the fund. The charge to operations on account of depreciation increases annually, but this increase is offset by the credit to interest, making the net expense the same each year.

6. Concerning the production method Montgomery says:

> A method of making depreciation allowances which has its advantages under certain conditions is that of charging an established rate per unit of

output. This is especially applicable in the case, say, of a blast furnace where the frequency with which the linings will need to be renewed depends on the extent to which the furnace is being used. If it is being run at full capacity night and day, the wear on the linings is obviously much greater than if the furnace had not been in continual use during the entire fiscal period.[2]

Mr. Montgomery applies this method with perfect justice to the lessening in value of a wasting asset, such as timber, or the coal or ore in a mine, but this lessening in value is not caused by depreciation, but by an actual consumption, or removal and conversion of the asset. In fact, this is the only possible method to be applied to those assets which diminish in exact ratio to the amount used.

There can be no rules formulated for the determination of the amount to be written off against each unit of production. That is a matter that must be left to the judgment of the factory managers, guided by experience.

The following is a comparative table of depreciation charges, made according to the first five methods:

Year	Straight Line	Per Cent of Diminishing Value	Sum of Years' Digits	Annuity	Sinking Fund
1	$ 800	$2,076.00	$1,371.43	$ 988.14	$ 723.66
2	800	1,214.04	1,142.86	988.14	752.61
3	800	709.98	914.29	988.14	782.71
4	800	415.19	685.71	988.14	814.02
5	800	242.80	457.14	988.14	846.58
6	800	141.99	228.57	988.14	880.44
Totals.....	$4,800	$4,800.00	$4,800.00	$5,928.84	$4,800.02
Credit to interest....				$1,128.84	
Net.....				$4,800.00	

[2] R. H. Montgomery, Auditing Theory and Practice, 1919, p. 550.

The comparative carrying values in the same example under the five methods are as follows:

End of Year	Straight Line	Per Cent of Diminishing Value	Sum of Years' Digits	Annuity	Sinking Fund
	$5,000	$5,000.00	$5,000.00	$5,000.00	$5,000.00
1	4,200	2,924.00	3,628.57	4,311 86	4,276.34
2	3,400	1,709.96	2,485.71	3,582.43	3,523.73
3	2,600	999.98	1,571.42	2,809.24	2,741.02
4	1,800	584.79	885.71	1,989.65	1,927.00
5	1,000	341.99	428.57	1,120.89	1,080.42
6	200	200.00	200.00	200.00	199.98

APPENDIX A

VALUES OF FOREIGN COINS

Following is a list of foreign monetary units and their values, representing the pars of exchange, as estimated by the United States Director of the Mint.

Country	Legal Standard	Monetary Unit		Value in Terms of U. S. Money
Argentine Republic	Gold	Peso		$0.9648
Austria-Hungary	Gold	Krone		.2026
Belgium	Gold and silver	Franc		.1930
Bolivia	Gold	Boliviano		.3893
Brazil	Gold	Milreis		.5462
British Colonies in Australasia and Africa	Gold	Pound sterling		4.8665
Canada	Gold	Dollar		1.0000
Central American States:				
Costa Rica	Gold	Colon		.4653
British Honduras	Gold	Dollar		1.0000
Nicaragua	Gold	Peso		1.0000
Guatemala, Honduras	Silver	Peso		.4403
Salvador	Gold	Peso		.5000
Chile	Gold	Peso		.3650
China	Silver	Tael	Amoy	.7219
			Canton	.7197
			Cheefoo	.6904
			Chin Kiang	.7052
			Fuchau	.6678
			Haikwan (customs)	.7345
			Hankow	.6754
			Kiaochow	.6995
			Nankin	.7143
			Niuchwang	.6770
			Ningpo	.6940
			Peking	.7037
			Shanghai	.6594
			Swatow	.6668
			Takau	.7264
			Tientsin	.6995
		Dollar	Yuan	.4730
			Hongkong	.4748
			British	.4748
			Mexican	.4783

Country	Legal Standrd	Monetary Unit	Value in Terms of U. S. Money
Colombia	Gold	(Dollar) Peso	.0733
Cuba	Gold	Dollar	1.0000
Denmark	Gold	Krone	.2680
Ecuador	Gold	Sucre	.4867
Egypt	Gold	Pound (100 piasters)	4.9431
Finland	Gold	Finmark	.1930
France	Gold and silver	Franc	.1930
Germany	Gold	Mark	.2382
Great Britain	Gold	Pound sterling	4.8665
Greece	Gold and silver	Drachma	.1930
Haiti	Gold	Gourde	.2500
India [British]	Gold	Rupee	.3244
Indo-China	Silver	Piaster	.4755
Italy	Gold	Lira	.1930
Japan	Gold	Yen	.4985
Liberia	Gold	Dollar	1.0000
Mexico	Gold	Peso	.4985
Netherlands	Gold	Guilder (Florin)	.4020
Newfoundland	Gold	Dollar	1.0000
Norway	Gold	Krone	.2680
Panama	Gold	Dollar	1.0000
Paraguay	Gold	Peso (Argentine)	.9648
Persia	Gold	Ashrafi	.0959
	Silver	Kran	.0811
Peru	Gold	Libra	4.8665
Philippine Islands	Gold	Peso	.5000
Portugal	Gold	Escudo	1.0805
Roumania	Gold	Leu	.1930
Russia	Gold	Ruble	.5146
Santo Domingo	Gold	Dollar	1.0000
Serbia	Gold	Dinar	.1930
Siam	Gold	Tical	.3709
Spain	Gold and silver	Peseta	.1930
Straits Settlements	Gold	Dollar	.5678
Sweden	Gold	Krona	.2680
Switzerland	Gold	Franc	.1930
Turkey	Gold	Turkish Pound	.0440
Uruguay	Gold	Peso	1.0342
Venezuela	Gold	Bolivar	.1930

APPENDIX B

LOGARITHMS OF NUMBERS[1]

No.	0	1	2	3	4	5	6	7	8	9
100	00 000	00 043	00 087	00 130	00 173	00 217	00 260	00 303	00 346	00 389
101	00 432	00 475	00 518	00 561	00 604	00 647	00 689	00 732	00 775	00 817
102	00 860	00 903	00 945	00 988	01 030	01 072	01 115	01 157	01 199	01 242
103	01 284	01 326	01 368	01 410	01 452	01 494	01 536	01 578	01 620	01 662
104	01 703	01 745	01 787	01 828	01 870	01 912	01 953	01 995	02 036	02 078
105	02 119	02 160	02 202	02 243	02 284	02 325	02 366	02 407	02 449	02 490
106	02 531	02 572	02 612	02 653	02 694	02 735	02 776	02 816	02 857	02 898
107	02 938	02 979	03 019	03 060	03 100	03 141	03 181	03 222	03 262	03 302
108	03 342	03 383	03 423	03 463	03 503	03 543	03 583	03 623	03 663	03 703
109	03 743	03 782	03 822	03 862	03 902	03 941	03 981	04 021	04 060	04 100
110	04 139	04 179	04 218	04 258	04 297	04 336	04 376	04 415	04 454	04 493
111	04 532	04 571	04 610	04 650	04 689	04 727	04 766	04 805	04 844	04 883
112	04 922	04 961	04 999	05 038	05 077	05 115	05 154	05 192	05 231	05 269
113	05 308	05 346	05 385	05 423	05 461	05 500	05 538	05 576	05 614	05 652
114	05 690	05 729	05 767	05 805	05 843	05 881	05 918	05 956	05 994	06 032
115	06 070	06 108	06 145	06 183	06 221	06 258	06 296	06 333	06 371	06 408
116	06 446	06 483	06 521	06 558	06 595	06 633	06 670	06 707	06 744	06 781
117	06 819	06 856	06 893	06 930	06 967	07 004	07 041	07 078	07 115	07 151
118	07 188	07 225	07 262	07 298	07 335	07 372	07 408	07 445	07 482	07 518
119	07 555	07 591	07 628	07 664	07 700	07 737	07 773	07 809	07 846	07 882
120	07 918	07 954	07 990	08 027	08 063	08 099	08 135	08 171	08 207	08 243
121	08 279	08 314	08 350	08 386	08 422	08 458	08 493	08 529	08 565	08 600
122	08 636	08 672	08 707	08 743	08 778	08 814	08 849	08 884	08 920	08 955
123	08 991	09 026	09 061	09 096	09 132	09 167	09 202	09 237	09 272	09 307
124	09 342	09 377	09 412	09 447	09 482	09 517	09 552	09 587	09 621	09 656
125	09 691	09 726	09 760	09 795	09 830	09 864	09 899	09 934	09 968	10 003
126	10 037	10 072	10 106	10 140	10 175	10 209	10 243	10 278	10 312	10 346
127	10 380	10 415	10 449	10 483	10 517	10 551	10 585	10 619	10 653	10 687
128	10 721	10 755	10 789	10 823	10 857	10 890	10 924	10 958	10 992	11 025
129	11 059	11 093	11 126	11 160	11 193	11 227	11 261	11 294	11 327	11 361
130	11 394	11 428	11 461	11 494	11 528	11 561	11 594	11 628	11 661	11 694
131	11 727	11 760	11 793	11 826	11 860	11 893	11 926	11 959	11 992	12 024
132	12 057	12 090	12 123	12 156	12 189	12 222	12 254	12 287	12 320	12 352
133	12 385	12 418	12 450	12 483	12 516	12 548	12 581	12 613	12 646	12 678
134	12 710	12 743	12 775	12 808	12 840	12 872	12 905	12 937	12 969	13 001

[1] E. H Barker, Computing Tables and Formulas, 1913, pages 22–39.

No.	0	1	2	3	4	5	6	7	8	9
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142	15 229	15 259	15 290	15 320	15 351	15 381	15 412	15 442	15 473	15 503
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156	19 312	19 340	19 368	19 396	19 424	19 451	19 479	19 507	19 535	19 562
157	19 590	19 618	19 645	19 673	19 700	19 728	19 756	19 783	19 811	19 838
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159	20 140	20 167	20 194	20 222	20 249	20 276	20 303	20 330	20 358	20 385
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178	25 042	25 066	25 091	25 115	25 139	25 164	25 188	25 212	25 237	25 261
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184	26 482	26 505	26 529	26 553	26 576	26 600	26 623	26 647	26 670	26 694
185	26 717	26 741	26 764	26 788	26 811	26 834	26 858	26 881	26 905	26 928
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187	27 184	27 207	27 231	27 254	27 277	27 300	27 323	27 346	27 370	27 393
188	27 416	27 439	27 462	27 485	27 508	27 531	27 554	27 577	27 600	27 623
189	27 646	27 669	27 692	27 715	27 738	27 761	27 784	27 807	27 830	27 852
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191	28 103	28 126	28 149	28 171	28 194	28 217	28 240	28 262	28 285	28 307
192	28 330	28 353	28 375	28 398	28 421	28 443	28 466	28 488	28 511	28 533
193	28 556	28 578	28 601	28 623	28 646	28 668	28 691	28 713	28 735	28 758
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201	30 320	30 341	30 363	30 384	30 406	30 428	30 449	30 471	30 492	30 514
202	30 535	30 557	30 578	30 600	30 621	30 643	30 664	30 685	30 707	30 728
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213	32 838	32 858	32 879	32 899	32 919	32 940	32 960	32 980	33 001	33 021
214	33 041	33 062	33 082	33 102	33 122	33 143	33 163	33 183	33 203	33 224
215	33 244	33 264	33 284	33 304	33 325	33 345	33 365	33 385	33 405	33 425
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217	33 646	33 666	33 686	33 706	33 726	33 746	33 766	33 786	33 806	33 826
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220	34 242	34 262	34 282	34 301	34 321	34 341	34 361	34 380	34 400	34 420
221	34 439	34 459	34 479	34 498	34 518	34 537	34 557	34 577	34 596	34 616
222	34 635	34 655	34 674	34 694	34 713	34 733	34 753	34 772	34 792	34 811
223	34 830	34 850	34 869	34 889	34 908	34 928	34 947	34 967	34 986	35 005
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247	39 270	39 287	39 305	39 322	39 340	39 358	39 375	39 393	39 410	39 428
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249	39 620	39 637	39 655	39 672	39 690	39 707	39 724	39 742	39 759	39 777
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266	42 488	42 504	42 521	42 537	42 553	42 570	42 586	42 602	42 619	42 635
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340	53 148	53 161	53 173	53 186	53 199	53 212	53 224	53 237	53 250	53 263
341	53 275	53 288	53 301	53 314	53 326	53 339	53 352	53 364	53 377	53 390
342	53 403	53 415	53 428	53 441	53 453	53 466	53 479	53 491	53 504	53 517
343	53 529	53 542	53 555	53 567	53 580	53 593	53 605	53 618	53 631	53 643
344	53 656	53 668	53 681	53 694	53 706	53 719	53 732	53 744	53 757	53 769
345	53 782	53 794	53 807	53 820	53 832	53 845	53 857	53 870	53 882	53 895
346	53 908	53 920	53 933	53 945	53 958	53 970	53 983	53 995	54 008	54 020
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349	54 283	54 295	54 307	54 320	54 332	54 345	54 357	54 370	54 382	54 394
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351	54 531	54 543	54 555	54 568	54 580	54 593	54 605	54 617	54 630	54 642
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356	55 145	55 157	55 169	55 182	55 194	55 206	55 218	55 230	55 242	55 255
357	55 267	55 279	55 291	55 303	55 315	55 328	55 340	55 352	55 364	55 376
358	55 388	55 400	55 413	55 425	55 437	55 449	55 461	55 473	55 485	55 497
359	55 509	55 522	55 534	55 540	55 558	55 570	55 582	55 594	55 606	55 618

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362	55 871	55 883	55 895	55 907	55 919	55 931	55 943	55 955	55 967	55 979
363	55 991	56 003	56 015	56 027	56 038	56 050	56 062	56 074	56 086	56 098
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366	56 348	56 360	56 372	56 384	56 396	56 407	56 419	56 431	56 443	56 455
367	56 467	56 478	56 490	56 502	56 514	56 526	56 538	56 549	56 561	56 573
368	56 585	56 597	56 608	56 620	56 632	56 644	56 656	56 667	56 679	56 691
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373	57 171	57 183	57 194	57 206	57 217	57 229	57 241	57 252	57 264	57 276
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375	57 403	57 415	57 426	57 438	57 449	57 461	57 473	57 484	57 496	57 507
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380	57 978	57 990	58 001	58 013	58 024	58 035	58 047	58 058	58 070	58 081
381	58 092	58 104	58 115	58 127	58 138	58 149	58 161	58 172	58 184	58 195
382	58 206	58 218	58 229	58 240	58 252	58 263	58 274	58 286	58 297	58 309
383	58 320	58 331	58 343	58 354	58 365	58 377	58 388	58 399	58 410	58 422
384	58 433	58 444	58 456	58 467	58 478	58 490	58 501	58 512	58 524	58 535
385	58 546	58 557	58 569	58 580	58 591	58 602	58 614	58 625	58 636	58 647
386	58 659	58 670	58 681	58 692	58 704	58 715	58 726	58 737	58 749	58 760
387	58 771	58 782	58 794	58 805	58 816	58 827	58 838	58 850	58 861	58 872
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391	59 218	59 229	59 240	59 251	59 262	59 273	59 284	59 295	59 306	59 318
392	59 329	59 340	59 351	59 362	59 373	59 384	59 395	59 406	59 417	59 428
393	59 439	59 450	59 461	59 472	59 483	59 494	59 506	59 517	59 528	59 539
394	59 550	59 561	59 572	59 583	59 594	59 605	59 616	59 627	59 638	59 649
395	59 660	59 671	59 682	59 693	59 704	59 715	59 726	59 737	59 748	59 759
396	59 770	59 780	59 791	59 802	59 813	59 824	59 835	59 846	59 857	59 868
397	59 879	59 890	59 901	59 912	59 923	59 934	59 945	59 956	59 966	59 977
398	59 988	59 999	60 010	60 021	60 032	60 043	60 054	60 065	60 076	60 086
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400	60 206	60 217	60 228	60 239	60 249	60 260	60 271	60 282	60 293	60 304
401	60 314	60 325	60 336	60 347	60 358	60 369	60 379	60 390	60 401	60 412
402	60 423	60 433	60 444	60 455	60 466	60 477	60 487	60 498	60 509	60 520
403	60 531	60 541	60 552	60 563	60 574	60 584	60 595	60 606	60 617	60 627
404	60 638	60 649	60 660	60 670	60 681	60 692	60 703	60 713	60 724	60 735

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409	61 172	61 183	61 194	61 204	61 215	61 225	61 236	61 247	61 257	61 268
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411	61 384	61 395	61 405	61 416	61 426	61 437	61 448	61 458	61 469	61 479
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414	61 700	61 711	61 721	61 731	61 742	61 752	61 763	61 773	61 784	61 794
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416	61 909	61 920	61 930	61 941	61 951	61 962	61 972	61 982	61 993	62 003
417	62 014	62 024	62 034	62 045	62 055	62 066	62 076	62 086	62 097	62 107
418	62 118	62 128	62 138	62 149	62 159	62 170	62 180	62 190	62 201	62 211
419	62 221	62 232	62 242	62 252	62 263	62 273	62 284	62 294	62 304	62 315
420	62 325	62 335	62 346	62 356	62 366	62 377	62 387	62 397	62 408	62 418
421	62 428	62 439	62 449	62 459	62 469	62 480	62 490	62 500	62 511	62 521
422	62 531	62 542	62 552	62 562	62 572	62 583	62 593	62 603	62 613	62 624
423	62 634	62 644	62 655	62 665	62 675	62 685	62 696	62 706	62 716	62 726
424	62 737	62 747	62 757	62 767	62 778	62 788	62 798	62 808	62 818	62 829
425	62 839	62 849	62 859	62 870	62 880	62 890	62 900	62 910	62 921	62 931
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428	63 144	63 155	63 165	63 175	63 185	63 195	63 205	63 215	63 225	63 236
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430	63 347	63 357	63 367	63 377	63 387	63 397	63 407	63 417	63 428	63 438
431	63 448	63 458	63 468	63 478	63 488	63 498	63 508	63 518	63 528	63 538
432	63 548	63 558	63 568	63 579	63 589	63 599	63 609	63 619	63 629	63 639
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436	63 949	63 959	63 969	63 979	63 988	63 998	64 008	64 018	64 028	64 038
437	64 048	64 058	64 068	64 078	64 088	64 098	64 108	64 118	64 128	64 137
438	64 147	64 157	64 167	64 177	64 187	64 197	64 207	64 217	64 227	64 237
439	64 246	64 256	64 266	64 276	64 286	64 296	64 306	64 316	64 326	64 335
440	64 345	64 355	64 365	64 375	64 385	64 395	64 404	64 414	64 424	64 434
441	64 444	64 454	64 464	64 473	64 483	64 493	64 503	64 513	64 523	64 532
442	64 542	64 552	64 562	64 572	64 582	64 591	64 601	64 611	64 621	64 631
443	64 640	64 650	64 660	64 670	64 680	64 689	64 699	64 709	64 719	64 729
444	64 738	64 748	64 758	64 768	64 777	64 787	64 797	64 807	64 816	64 826
445	64 836	64 846	64 856	64 865	64 875	64 885	64 895	64 904	64 914	64 924
446	64 933	64 943	64 953	64 963	64 972	64 982	64 992	65 002	65 011	65 021
447	65 031	65 040	65 050	65 060	65 070	65 079	65 089	65 099	65 108	65 118
448	65 128	65 137	65 147	65 157	65 167	65 176	65 186	65 196	65 205	65 215
449	65 225	65 234	65 244	65 254	65 263	65 273	65 283	65 292	65 302	65 312

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452	65 514	65 523	65 533	65 543	65 552	65 562	65 571	65 581	65 591	65 600
453	65 610	65 619	65 629	65 639	65 648	65 658	65 667	65 677	65 686	65 696
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455	65 801	65 811	65 820	65 830	65 839	65 849	65 858	65 868	65 877	65 887
456	65 896	65 906	65 916	65 925	65 935	65 944	65 954	65 963	65 973	65 982
457	65 992	66 001	66 011	66 020	66 030	66 039	66 049	66 058	66 068	66 077
458	66 087	66 096	66 106	66 115	66 124	66 134	66 143	66 153	66 162	66 172
459	66 181	66 191	66 200	66 210	66 219	66 229	66 238	66 247	66 257	66 266
460	66 276	66 285	66 295	66 304	66 314	66 323	66 332	66 342	66 351	66 361
461	66 370	66 380	66 389	66 398	66 408	66 417	66 427	66 436	66 445	66 455
462	66 464	66 474	66 483	66 492	66 502	66 511	66 521	66 530	66 539	66 549
463	66 558	66 567	66 577	66 586	66 596	66 605	66 614	66 624	66 633	66 642
464	66 652	66 661	66 671	66 680	66 689	66 699	66 708	66 717	66 727	66 736
465	66 745	66 755	66 764	66 773	66 783	66 792	66 801	66 811	66 820	66 829
466	66 839	66 848	66 857	66 867	66 876	66 885	66 894	66 904	66 913	66 922
467	66 932	66 941	66 950	66 960	66 969	66 978	66 987	66 997	67 006	67 015
468	67 025	67 034	67 043	67 052	67 062	67 071	67 080	67 089	67 099	67 108
469	67 117	67 127	67 136	67 145	67 154	67 164	67 173	67 182	67 191	67 201
470	67 210	67 219	67 228	67 237	67 247	67 256	67 265	67 274	67 284	67 293
471	67 302	67 311	67 321	67 330	67 339	67 348	67 357	67 367	67 376	67 385
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476	67 761	67 770	67 779	67 788	67 797	67 806	67 815	67 825	67 834	67 843
477	67 852	67 861	67 870	67 879	67 888	67 897	67 906	67 916	67 925	67 934
478	67 943	67 952	67 961	67 970	67 979	67 988	67 997	68 006	68 015	68 024
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484	68 485	68 494	68 502	68 511	68 520	68 529	68 538	68 547	68 556	68 565
485	68 574	68 583	68 592	68 601	68 610	68 619	68 628	68 637	68 646	68 655
486	68 664	68 673	68 681	68 690	68 699	68 708	68 717	68 726	68 735	68 744
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488	68 842	68 851	68 860	68 869	68 878	68 886	68 895	68 904	68 913	68 922
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490	69 020	69 028	69 037	69 046	69 055	69 064	69 073	69 082	69 090	69 099
491	69 108	69 117	69 126	69 135	69 144	69 152	69 161	69 170	69 179	69 188
492	69 197	69 205	69 214	69 223	69 232	69 241	69 249	69 258	69 267	69 276
493	69 285	69 294	69 302	69 311	69 320	69 329	69 338	69 346	69 355	69 364
494	69 373	69 381	69 390	69 399	69 408	69 417	69 425	69 434	69 443	69 452

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497	69 636	69 644	69 653	69 662	69 671	69 679	69 688	69 697	69 705	69 714
498	69 723	69 732	69 740	69 749	69 758	69 767	69 775	69 784	69 793	69 801
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504	70 243	70 252	70 260	70 269	70 278	70 286	70 295	70 303	70 312	70 321
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506	70 415	70 424	70 432	70 441	70 449	70 458	70 467	70 475	70 484	70 492
507	70 501	70 509	70 518	70 526	70 535	70 544	70 552	70 561	70 569	70 578
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524	71 933	71 941	71 950	71 958	71 966	71 975	71 983	71 991	71 999	72 008
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527	72 181	72 189	72 198	72 206	72 214	72 222	72 230	72 239	72 247	72 255
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531	72 509	72 518	72 526	72 534	72 542	72 550	72 558	72 567	72 575	72 583
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533	72 673	72 681	72 689	72 697	72 705	72 713	72 722	72 730	72 738	72 746
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535	72 835	72 843	72 852	72 860	72 868	72 876	72 884	72 892	72 900	72 908
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538	73 078	73 086	73 094	73 102	73 111	73 119	73 127	73 135	73 143	73 151
539	73 159	73 167	73 175	73 183	73 191	73 199	73 207	73 215	73 223	73 231

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542	73 400	73 408	73 416	73 424	73 432	73 440	73 448	73 456	73 464	73 472
543	73 480	73 488	73 496	73 504	73 512	73 520	73 528	73 536	73 544	73 552
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546	73 719	73 727	73 735	73 743	73 751	73 759	73 767	73 775	73 783	73 791
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549	73 957	73 965	73 973	73 981	73 989	73 997	74 005	74 013	74 020	74 028
550	74 036	74 044	74 052	74 060	74 068	74 076	74 084	74 092	74 099	74 107
551	74 115	74 123	74 131	74 139	74 147	74 155	74 162	74 170	74 178	74 186
552	74 194	74 202	74 210	74 218	74 225	74 233	74 241	74 249	74 257	74 265
553	74 273	74 280	74 288	74 296	74 304	74 312	74 320	74 327	74 335	74 343
554	74 351	74 359	74 367	74 374	74 382	74 390	74 398	74 406	74 414	74 421
555	74 429	74 437	74 445	74 453	74 461	74 468	74 476	74 484	74 492	74 500
556	74 507	74 515	74 523	74 531	74 539	74 547	74 554	74 562	74 570	74 578
557	74 586	74 593	74 601	74 609	74 617	74 624	74 632	74 640	74 648	74 656
558	74 663	74 671	74 679	74 687	74 695	74 702	74 710	74 718	74 726	74 733
559	74 741	74 749	74 757	74 764	74 772	74 780	74 788	74 796	74 803	74 811
560	74 819	74 827	74 834	74 842	74 850	74 858	74 865	74 873	74 881	74 889
561	74 896	74 904	74 912	74 920	74 927	74 935	74 943	74 950	74 958	74 966
562	74 974	74 981	74 989	74 997	75 005	75 012	75 020	75 028	75 035	75 043
563	75 051	75 059	75 066	75 074	75 082	75 089	75 097	75 105	75 113	75 120
564	75 128	75 136	75 143	75 151	75 159	75 166	75 174	75 182	75 189	75 197
565	75 205	75 213	75 220	75 228	75 236	75 243	75 251	75 259	75 266	75 274
566	75 282	75 289	75 297	75 305	75 312	75 320	75 328	75 335	75 343	75 351
567	75 358	75 366	75 374	75 381	75 389	75 397	75 404	75 412	75 420	75 427
568	75 435	75 442	75 450	75 458	75 465	75 473	75 481	75 488	75 496	75 504
569	75 511	75 519	75 526	75 534	75 542	75 549	75 557	75 565	75 572	75 580
570	75 587	75 595	75 603	75 610	75 618	75 626	75 633	75 641	75 648	75 656
571	75 664	75 671	75 679	75 686	75 694	75 702	75 709	75 717	75 724	75 732
572	75 740	75 747	75 755	75 762	75 770	75 778	75 785	75 793	75 800	75 808
573	75 815	75 823	75 831	75 838	75 846	75 853	75 861	75 868	75 876	75 884
574	75 891	75 899	75 906	75 914	75 921	75 929	75 937	75 944	75 952	75 959
575	75 967	75 974	75 982	75 989	75 997	76 005	76 012	76 020	76 027	76 035
576	76 042	76 050	76 057	76 065	76 072	76 080	76 087	76 095	76 103	76 110
577	76 118	76 125	76 133	76 140	76 148	76 155	76 163	76 170	76 178	76 185
578	76 193	76 200	76 208	76 215	76 223	76 230	76 238	76 245	76 253	76 260
579	76 268	76 275	76 283	76 290	76 298	76 305	76 313	76 320	76 328	76 335
580	76 343	76 350	76 358	76 365	76 373	76 380	76 388	76 395	76 403	76 410
581	76 418	76 425	76 433	76 440	76 448	76 455	76 462	76 470	76 477	76 485
582	76 492	76 500	76 507	76 515	76 522	76 530	76 537	76 545	76 552	76 559
583	76 567	76 574	76 582	76 589	76 597	76 604	76 612	76 619	76 626	76 634
584	76 641	76 649	76 656	76 664	76 671	76 678	76 686	76 693	76 701	76 708

No.	0	1	2	3	4	5	6	7	8	9
585	76 716	76 723	76 730	76 738	76 745	76 753	76 760	76 768	76 775	76 782
586	76 790	76 797	76 805	76 812	76 819	76 827	76 834	76 842	76 849	76 856
587	76 864	76 871	76 879	76 886	76 893	76 901	76 908	76 916	76 923	76 930
588	76 938	76 945	76 953	76 960	76 967	76 975	76 982	76 989	76 997	77 004
589	77 012	77 019	77 026	77 034	77 041	77 048	77 056	77 063	77 070	77 078
590	77 085	77 093	77 100	77 107	77 115	77 122	77 129	77 137	77 144	77 151
591	77 159	77 166	77 173	77 181	77 188	77 195	77 203	77 210	77 217	77 225
592	77 232	77 240	77 247	77 254	77 262	77 269	77 276	77 283	77 291	77 298
593	77 305	77 313	77 320	77 327	77 335	77 342	77 349	77 357	77 364	77 371
594	77 379	77 386	77 393	77 401	77 408	77 415	77 422	77 430	77 437	77 444
595	77 452	77 459	77 466	77 474	77 481	77 488	77 495	77 503	77 510	77 517
596	77 525	77 532	77 539	77 546	77 554	77 561	77 568	77 576	77 583	77 590
597	77 597	77 605	77 612	77 619	77 627	77 634	77 641	77 648	77 656	77 663
598	77 670	77 677	77 685	77 692	77 699	77 706	77 714	77 721	77 728	77 735
599	77 743	77 750	77 757	77 764	77 772	77 779	77 786	77 793	77 801	77 808
600	77 815	77 822	77 830	77 837	77 844	77 851	77 859	77 866	77 873	77 880
601	77 887	77 895	77 902	77 909	77 916	77 924	77 931	77 938	77 945	77 952
602	77 960	77 967	77 974	77 981	77 988	77 996	78 003	78 010	78 017	78 025
603	78 032	78 039	78 046	78 053	78 061	78 068	78 075	78 082	78 089	78 097
604	78 104	78 111	78 118	78 125	78 132	78 140	78 147	78 154	78 161	78 168
605	78 176	78 183	78 190	78 197	78 204	78 211	78 219	78 226	78 233	78 240
606	78 247	78 254	78 262	78 269	78 276	78 283	78 290	78 297	78 305	78 312
607	78 319	78 326	78 333	78 340	78 347	78 355	78 362	78 369	78 376	78 383
608	78 390	78 398	78 405	78 412	78 419	78 426	78 433	78 440	78 447	78 455
609	78 462	78 469	78 476	78 483	78 490	78 497	78 504	78 512	78 519	78 526
610	78 533	78 540	78 547	78 554	78 561	78 569	78 576	78 583	78 590	78 597
611	78 604	78 611	78 618	78 625	78 633	78 640	78 647	78 654	78 661	78 668
612	78 675	78 682	78 689	78 696	78 704	78 711	78 718	78 725	78 732	78 739
613	78 746	78 753	78 760	78 767	78 774	78 781	78 789	78 796	78 803	78 810
614	78 817	78 824	78 831	78 838	78 845	78 852	78 859	78 866	78 873	78 880
615	78 888	78 895	78 902	78 909	78 916	78 923	78 930	78 937	78 944	78 951
616	78 958	78 965	78 972	78 979	78 986	78 993	79 000	79 007	79 014	79 021
617	79 029	79 036	79 043	79 050	79 057	79 064	79 071	79 078	79 085	79 092
618	79 099	79 106	79 113	79 120	79 127	79 134	79 141	79 148	79 155	79 162
619	79 169	79 176	79 183	79 190	79 197	79 204	79 211	79 218	79 225	79 232
620	79 239	79 246	79 253	79 260	79 267	79 274	79 281	79 288	79 295	79 302
621	79 309	79 316	79 323	79 330	79 337	79 344	79 351	79 358	79 365	79 372
622	79 379	79 386	79 393	79 400	79 407	79 414	79 421	79 428	79 435	79 442
623	79 449	79 456	79 463	79 470	79 477	79 484	79 491	79 498	79 505	79 511
624	79 518	79 525	79 532	79 539	79 546	79 553	79 560	79 567	79 574	79 581
625	79 588	79 595	79 602	79 609	79 616	79 623	79 630	79 637	79 644	79 650
626	79 657	79 664	79 671	79 678	79 685	79 692	79 699	79 706	79 713	79 720
627	79 727	79 734	79 741	79 748	79 754	79 761	79 768	79 775	79 782	79 789
628	79 796	79 803	79 810	79 817	79 824	79 831	79 837	79 844	79 851	79 858
629	79 865	79 872	79 879	79 886	79 893	79 900	79 906	79 913	79 920	79 927

No.	0	1	2	3	4	5	6	7	8	9
630	79 934	79 941	79 948	79 955	79 962	79 969	79 975	79 982	79 989	79 996
631	80 003	80 010	80 017	80 024	80 030	80 037	80 044	80 051	80 058	80 065
632	80 072	80 079	80 085	80 092	80 099	80 106	80 113	80 120	80 127	80 134
633	80 140	80 147	80 154	80 161	80 168	80 175	80 182	80 188	80 195	80 202
634	80 209	80 216	80 223	80 229	80 236	80 243	80 250	80 257	80 264	80 271
635	80 277	80 284	80 291	80 298	80 305	80 312	80 318	80 325	80 332	80 339
636	80 346	80 353	80 359	80 366	80 373	80 380	80 387	80 393	80 400	80 407
637	80 414	80 421	80 428	80 434	80 441	80 448	80 455	80 462	80 468	80 475
638	80 482	80 489	80 496	80 502	80 509	80 516	80 523	80 530	80 536	80 543
639	80 550	80 557	80 564	80 570	80 577	80 584	80 591	80 598	80 604	80 611
640	80 618	80 625	80 632	80 638	80 645	80 652	80 659	80 665	80 672	80 679
641	80 686	80 693	80 699	80 706	80 713	80 720	80 726	80 733	80 740	80 747
642	80 754	80 760	80 767	80 774	80 781	80 787	80 794	80 801	80 808	80 814
643	80 821	80 828	80 835	80 841	80 848	80 855	80 862	80 868	80 875	80 882
644	80 889	80 895	80 902	80 909	80 916	80 922	80 929	80 936	80 943	80 949
645	80 956	80 963	80 969	80 976	80 983	80 990	80 996	81 003	81 010	81 017
646	81 023	81 030	81 037	81 043	81 050	81 057	81 064	81 070	81 077	81 084
647	81 090	81 097	81 104	81 111	81 117	81 124	81 131	81 137	81 144	81 151
648	81 158	81 164	81 171	81 178	81 184	81 191	81 198	81 204	81 211	81 218
649	81 224	81 231	81 238	81 245	81 251	81 258	81 265	81 271	81 278	81 285
650	81 291	81 298	81 305	81 311	81 318	81 325	81 331	81 338	81 345	81 351
651	81 358	81 365	81 371	81 378	81 385	81 391	81 398	81 405	81 411	81 418
652	81 425	81 431	81 438	81 445	81 451	81 458	81 465	81 471	81 478	81 485
653	81 491	81 498	81 505	81 511	81 518	81 525	81 531	81 538	81 544	81 551
654	81 558	81 564	81 571	81 578	81 584	81 591	81 598	81 604	81 611	81 617
655	81 624	81 631	81 637	81 644	81 651	81 657	81 664	81 671	81 677	81 684
656	81 690	81 697	81 704	81 710	81 717	81 723	81 730	81 737	81 743	81 750
657	81 757	81 763	81 770	81 776	81 783	81 790	81 796	81 803	81 809	81 816
658	81 823	81 829	81 836	81 842	81 849	81 856	81 862	81 869	81 875	81 882
659	81 889	81 895	81 902	81 908	81 915	81 921	81 928	81 935	81 941	81 948
660	81 954	81 961	81 968	81 974	81 981	81 987	81 994	82 000	82 007	82 014
661	82 020	82 027	82 033	82 040	82 046	82 053	82 060	82 066	82 073	82 079
662	82 086	82 092	82 099	82 105	82 112	82 119	82 125	82 132	82 138	82 145
663	82 151	82 158	82 164	82 171	82 178	82 184	82 191	82 197	82 204	82 210
664	82 217	82 223	82 230	82 236	82 243	82 249	82 256	82 263	82 269	82 276
665	82 282	82 289	82 295	82 302	82 308	82 315	82 321	82 328	82 334	82 341
666	82 347	82 354	82 360	82 367	82 373	82 380	82 387	82 393	82 400	82 406
667	82 413	82 419	82 426	82 432	82 439	82 445	82 452	82 458	82 465	82 471
668	82 478	82 484	82 491	82 497	82 504	82 510	82 517	82 523	82 530	82 536
669	82 543	82 549	82 556	82 562	82 569	82 575	82 582	82 588	82 595	82 601
670	82 607	82 614	82 620	82 627	82 633	82 640	82 646	82 653	82 659	82 666
671	82 672	82 679	82 685	82 692	82 698	82 705	82 711	82 718	82 724	82 730
672	82 737	82 743	82 750	82 756	82 763	82 769	82 776	82 782	82 789	82 795
673	82 802	82 808	82 814	82 821	82 827	82 834	82 840	82 847	82 853	82 860
674	82 866	82 872	82 879	82 885	82 892	82 898	82 905	82 911	82 918	82 924

No.	0	1	2	3	4	5	6	7	8	9
675	82 930	82 937	82 943	82 950	82 956	82 963	82 969	82 975	82 982	82 988
676	82 995	83 001	83 008	83 014	83 020	83 027	83 033	83 040	83 046	83 052
677	83 059	83 065	83 072	83 078	83 085	83 091	83 097	83 104	83 110	83 117
678	83 123	83 129	83 136	83 142	83 149	83 155	83 161	83 168	83 174	83 181
679	83 187	83 193	83 200	83 206	83 213	83 219	83 225	83 232	83 238	83 245
680	83 251	83 257	83 264	83 270	83 276	83 283	83 289	83 296	83 302	83 308
681	83 315	83 321	83 327	83 334	83 340	83 347	83 353	83 359	83 366	83 372
682	83 378	83 385	83 391	83 398	83 404	83 410	83 417	83 423	83 429	83 436
683	83 442	83 448	83 455	83 461	83 467	83 474	83 480	83 487	83 493	83 499
684	83 506	83 512	83 518	83 525	83 531	83 537	83 544	83 550	83 556	83 563
685	83 569	83 575	83 582	83 588	83 594	83 601	83 607	83 613	83 620	83 626
686	83 632	83 639	83 645	83 651	83 658	83 664	83 670	83 677	83 683	83 689
687	83 696	83 702	83 708	83 715	83 721	83 727	83 734	83 740	83 746	83 753
688	83 759	83 765	83 771	83 778	83 784	83 790	83 797	83 803	83 809	83 816
689	83 822	83 828	83 835	83 841	83 847	83 853	83 860	83 866	83 872	83 879
690	83 885	83 891	83 897	83 904	83 910	83 916	83 923	83 929	83 935	83 942
691	83 948	83 954	83 960	83 967	83 973	83 979	83 985	83 992	83 998	84 004
692	84 011	84 017	84 023	84 029	84 036	84 042	84 048	84 055	84 061	84 067
693	84 073	84 080	84 086	84 092	84 098	84 105	84 111	84 117	84 123	84 130
694	84 136	84 142	84 148	84 155	84 161	84 167	84 173	84 180	84 186	84 192
695	84 198	84 205	84 211	84 217	84 223	84 230	84 236	84 242	84 248	84 255
696	84 261	84 267	84 273	84 280	84 286	84 292	84 298	84 305	84 311	84 317
697	84 323	84 330	84 336	84 342	84 348	84 354	84 361	84 367	84 373	84 379
698	84 386	84 392	84 398	84 404	84 410	84 417	84 423	84 429	84 435	84 442
699	84 448	84 454	84 460	84 466	84 473	84 479	84 485	84 491	84 497	84 504
700	84 510	84 516	84 522	84 528	84 535	84 541	84 547	84 553	84 559	84 566
701	84 572	84 578	84 584	84 590	84 597	84 603	84 609	84 615	84 621	84 628
702	84 634	84 640	84 646	84 652	84 658	84 665	84 671	84 677	84 683	84 689
703	84 696	84 702	84 708	84 714	84 720	84 726	84 733	84 739	84 745	84 751
704	84 757	84 763	84 770	84 776	84 782	84 788	84 794	84 800	84 807	84 813
705	84 819	84 825	84 831	84 837	84 844	84 850	84 856	84 862	84 868	84 874
706	84 880	84 887	84 893	84 899	84 905	84 911	84 917	84 924	84 930	84 936
707	84 942	84 948	84 954	84 960	84 967	84 973	84 979	84 985	84 991	84 997
708	85 003	85 009	85 016	85 022	85 028	85 034	85 040	85 046	85 052	85 058
709	85 065	85 071	85 077	85 083	85 089	85 095	85 101	85 107	85 114	85 120
710	85 126	85 132	85 138	85 144	85 150	85 156	85 163	85 169	85 175	85 181
711	85 187	85 193	85 199	85 205	85 211	85 217	85 224	85 230	85 236	85 242
712	85 248	85 254	85 260	85 266	85 272	85 278	85 285	85 291	85 297	85 303
713	85 309	85 315	85 321	85 327	85 333	85 339	85 345	85 352	85 358	85 364
714	85 370	85 376	85 382	85 388	85 394	85 400	85 406	85 412	85 418	85 425
715	85 431	85 437	85 443	85 449	85 455	85 461	85 467	85 473	85 479	85 485
716	85 491	85 497	85 503	85 509	85 516	85 522	85 528	85 534	85 540	85 546
717	85 552	85 558	85 564	85 570	85 576	85 582	85 588	85 594	85 600	85 606
718	85 612	85 618	85 625	85 631	85 637	85 643	85 649	85 655	85 661	85 667
719	85 673	85 679	85 685	85 691	85 697	85 703	85 709	85 715	85 721	85 727

No.	0	1	2	3	4	5	6	7	8	9
720	85 733	85 739	85 745	85 751	85 757	85 763	85 769	85 775	85 781	85 788
721	85 794	85 800	85 806	85 812	85 818	85 824	85 830	85 836	85 842	85 848
722	85 854	85 860	85 866	85 872	85 878	85 884	85 890	85 896	85 902	85 908
723	85 914	85 920	85 926	85 932	85 938	85 944	85 950	85 956	85 962	85 968
724	85 974	85 980	85 986	85 992	85 998	86 004	86 010	86 016	86 022	86 028
725	86 034	86 040	86 046	86 052	86 058	86 064	86 070	86 076	86 082	86 088
726	86 094	86 100	86 106	86 112	86 118	86 124	86 130	86 136	86 141	86 147
727	86 153	86 159	86 165	86 171	86 177	86 183	86 189	86 195	86 201	86 207
728	86 213	86 219	86 225	86 231	86 237	86 243	86 249	86 255	86 261	86 267
729	86 273	86 279	86 285	86 291	86 297	86 303	86 308	86 314	86 320	86 326
730	86 332	86 338	86 344	86 350	86 356	86 362	86 368	86 374	86 380	86 386
731	86 392	86 398	86 404	86 410	86 415	86 421	86 427	86 433	86 439	86 445
732	86 451	86 457	86 463	86 469	86 475	86 481	86 487	86 493	86 499	86 504
733	86 510	86 516	86 522	86 528	86 534	86 540	86 546	86 552	86 558	86 564
734	86 570	86 576	86 581	86 587	86 593	86 599	86 605	86 611	86 617	86 623
735	86 629	86 635	86 641	86 646	86 652	86 658	86 664	86 670	86 676	86 682
736	86 688	86 694	86 700	86 705	86 711	86 717	86 723	86 729	86 735	86 741
737	86 747	86 753	86 759	86 764	86 770	86 776	86 782	86 788	86 794	86 800
738	86 806	86 812	86 817	86 823	86 829	86 835	86 841	86 847	86 853	86 859
739	86 864	86 870	86 876	86 882	86 888	86 894	86 900	86 906	86 911	86 917
740	86 923	86 929	86 935	86 941	86 947	86 953	86 958	86 964	86 970	86 976
741	86 982	86 988	86 994	86 999	87 005	87 011	87 017	87 023	87 029	87 035
742	87 040	87 046	87 052	87 058	87 064	87 070	87 075	87 081	87 087	87 093
743	87 099	87 105	87 111	87 116	87 122	87 128	87 134	87 140	87 146	87 151
744	87 157	87 163	87 169	87 175	87 181	87 186	87 192	87 198	87 204	87 210
745	87 216	87 221	87 227	87 233	87 239	87 245	87 251	87 256	87 262	87 268
746	87 274	87 280	87 286	87 291	87 297	87 303	87 309	87 315	87 320	87 326
747	87 332	87 338	87 344	87 349	87 355	87 361	87 367	87 373	87 379	87 384
748	87 390	87 396	87 402	87 408	87 413	87 419	87 425	87 431	87 437	87 442
749	87 448	87 454	87 460	87 466	87 471	87 477	87 483	87 489	87 495	87 500
750	87 506	87 512	87 518	87 523	87 529	87 535	87 541	87 547	87 552	87 558
751	87 564	87 570	87 576	87 581	87 587	87 593	87 599	87 604	87 610	87 616
752	87 622	87 628	87 633	87 639	87 645	87 651	87 656	87 662	87 668	87 674
753	87 679	87 685	87 691	87 697	87 703	87 708	87 714	87 720	87 726	87 731
754	87 737	87 743	87 749	87 754	87 760	87 766	87 772	87 777	87 783	87 789
755	87 795	87 800	87 806	87 812	87 818	87 823	87 829	87 835	87 841	87 846
756	87 852	87 858	87 864	87 869	87 875	87 881	87 887	87 892	87 898	87 904
757	87 910	87 915	87 921	87 927	87 933	87 938	87 944	87 950	87 955	87 961
758	87 967	87 973	87 978	87 984	87 990	87 996	88 001	88 007	88 013	88 018
759	88 024	88 030	88 036	88 041	88 047	88 053	88 058	88 064	88 070	88 076
760	88 081	88 087	88 093	88 098	88 104	88 110	88 116	88 121	88 127	88 133
761	88 138	88 144	88 150	88 156	88 161	88 167	88 173	88 178	88 184	88 190
762	88 195	88 201	88 207	88 213	88 218	88 224	88 230	88 235	88 241	88 247
763	88 252	88 258	88 264	88 270	88 275	88 281	88 287	88 292	88 298	88 304
764	88 309	88 315	88 321	88 326	88 332	88 338	88 343	88 349	88 355	88 360

No.	0	1	2	3	4	5	6	7	8	9
765	88 366	88 372	88 377	88 383	88 389	88 395	88 400	88 406	88 412	88 417
766	88 423	88 429	88 434	88 440	88 446	88 451	88 457	88 463	88 468	88 474
767	88 480	88 485	88 491	88 497	88 502	88 508	88 513	88 519	88 525	88 530
768	88 536	88 542	88 547	88 553	88 559	88 564	88 570	88 576	88 581	88 587
769	88 593	88 598	88 604	88 610	88 615	88 621	88 627	88 632	88 638	88 643
770	88 649	88 655	88 660	88 666	88 672	88 677	88 683	88 689	88 694	88 700
771	88 705	88 711	88 717	88 722	88 728	88 734	88 739	88 745	88 750	88 756
772	88 762	88 767	88 773	88 779	88 784	88 790	88 705	88 801	88 807	88 812
773	88 818	88 824	88 829	88 835	88 840	88 846	88 852	88 857	88 863	88 868
774	88 874	88 880	88 885	88 891	88 897	88 902	88 908	88 913	88 919	88 925
775	88 930	88 936	88 941	88 947	88 953	88 958	88 964	88 969	88 975	88 981
776	88 986	88 992	88 997	89 003	89 009	89 014	89 020	89 025	89 031	89 037
777	89 042	89 048	89 053	89 059	89 064	89 070	89 076	89 081	89 087	89 092
778	89 098	89 104	89 109	89 115	89 120	89 126	89 131	89 137	89 143	89 148
779	89 154	89 159	89 165	89 170	89 176	89 182	89 187	89 193	89 198	89 204
780	89 209	89 215	89 221	89 226	89 232	89 237	89 243	89 248	89 254	89 260
781	89 265	89 271	89 276	89 282	89 287	89 293	89 298	89 304	89 310	89 315
782	89 321	89 326	89 332	89 337	89 343	89 348	89 354	89 360	89 365	89 371
783	89 376	89 382	89 387	89 393	89 398	89 404	89 409	89 415	89 421	89 426
784	89 432	89 437	89 443	89 448	89 454	89 459	89 465	89 470	89 476	89 481
785	89 487	89 492	89 498	89 504	89 509	89 515	89 520	89 526	89 531	89 537
786	89 542	89 548	89 553	89 559	89 564	89 570	89 575	89 581	89 586	89 592
787	89 597	89 603	89 609	89 614	89 620	89 625	89 631	89 636	89 642	89 647
788	89 653	89 658	89 664	89 669	89 675	89 680	89 686	89 691	89 697	89 702
789	89 708	89 713	89 719	89 724	89 730	89 735	89 741	89 746	89 752	89 757
790	89 763	89 768	89 774	89 779	89 785	89 790	89 796	89 801	89 807	89 812
791	89 818	89 823	89 829	89 834	89 840	89 845	89 851	89 856	89 862	89 867
792	89 873	89 878	89 883	89 889	89 894	89 900	89 905	89 911	89 916	89 922
793	89 927	89 933	89 938	89 944	89 949	89 955	89 960	89 966	89 971	89 977
794	89 982	89 988	89 993	89 998	90 004	90 009	90 015	90 020	90 026	90 031
795	90 037	90 042	90 048	90 053	90 059	90 064	90 069	90 075	90 080	90 086
796	90 091	90 097	90 102	90 108	90 113	90 119	90 124	90 129	90 135	90 140
797	90 146	90 151	90 157	90 162	90 168	90 173	90 179	90 184	90 189	90 195
798	90 200	90 206	90 211	90 217	90 222	90 227	90 233	90 238	90 244	90 249
799	90 255	90 260	90 266	90 271	90 276	90 282	90 287	90 293	90 298	90 304
800	90 309	90 314	90 320	90 325	90 331	90 336	90 342	90 347	90 352	90 358
801	90 363	90 369	90 374	90 380	90 385	90 390	90 396	90 401	90 407	90 412
802	90 417	90 423	90 428	90 434	90 439	90 445	90 450	90 455	90 461	90 466
803	90 472	90 477	90 482	90 488	90 493	90 499	90 504	90 509	90 515	90 520
804	90 526	90 531	90 536	90 542	90 547	90 553	90 558	90 563	90 569	90 574
805	90 580	90 585	90 590	90 596	90 601	90 607	90 612	90 617	90 623	90 628
806	90 634	90 639	90 644	90 650	90 655	90 660	90 666	90 671	90 677	90 682
807	90 687	90 693	90 698	90 703	90 709	90 714	90 720	90 725	90 730	90 736
808	90 741	90 747	90 752	90 757	90 763	90 768	90 773	90 779	90 784	90 789
809	90 795	90 800	90 806	90 811	90 816	90 822	90 827	90 832	90 838	90 843

No.	0	1	2	3	4	5	6	7	8	9
810	90 849	90 854	90 859	90 865	90 870	90 875	90 881	90 886	90 891	90 897
811	90 902	90 907	90 913	90 918	90 924	90 929	90 934	90 940	90 945	90 950
812	90 956	90 961	90 966	90 972	90 977	90 982	90 988	90 993	90 998	91 004
813	91 009	91 014	91 020	91 025	91 030	91 036	91 041	91 046	91 052	91 057
814	91 062	91 068	91 073	91 078	91 084	91 089	91 094	91 100	91 105	91 110
815	91 116	91 121	91 126	91 132	91 137	91 142	91 148	91 153	91 158	91 164
816	91 169	91 174	91 180	91 185	91 190	91 196	91 201	91 206	91 212	91 217
817	91 222	91 228	91 233	91 238	91 243	91 249	91 254	91 259	91 265	91 270
818	91 275	91 281	91 286	91 291	91 297	91 302	91 307	91 312	91 318	91 323
819	91 328	91 334	91 339	91 344	91 350	91 355	91 360	91 365	91 371	91 376
820	91 381	91 387	91 392	91 397	91 403	91 408	91 413	91 418	91 424	91 429
821	91 434	91 440	91 445	91 450	91 455	91 461	91 466	91 471	91 477	91 482
822	91 487	91 492	91 498	91 503	91 508	91 514	91 519	91 524	91 529	91 535
823	91 540	91 545	91 551	91 556	91 561	91 566	91 572	91 577	91 582	91 587
824	91 593	91 598	91 603	91 609	91 614	91 619	91 624	91 630	91 635	91 640
825	91 645	91 651	91 656	91 661	91 666	91 672	91 677	91 682	91 687	91 693
826	91 698	91 703	91 709	91 714	91 719	91 724	91 730	91 735	91 740	91 745
827	91 751	91 756	91 761	91 766	91 772	91 777	91 782	91 787	91 793	91 798
828	91 803	91 808	91 814	91 819	91 824	91 829	91 834	91 840	91 845	91 850
829	91 855	91 861	91 866	91 871	91 876	91 882	91 887	91 892	91 897	91 903
830	91 908	91 913	91 918	91 924	91 929	91 934	91 939	91 944	91 950	91 955
831	91 960	91 965	91 971	91 976	91 981	91 986	91 991	91 997	92 002	92 007
832	92 012	92 018	92 023	92 028	92 033	92 038	92 044	92 049	92 054	92 059
833	92 065	92 070	92 075	92 080	92 085	92 091	92 096	92 101	92 106	92 111
834	92 117	92 122	92 127	92 132	92 137	92 143	92 148	92 153	92 158	92 163
835	92 169	92 174	92 179	92 184	92 189	92 195	92 200	92 205	92 210	92 215
836	92 221	92 226	92 231	92 236	92 241	92 247	92 252	92 257	92 262	92 267
837	92 273	92 278	92 283	92 288	92 293	92 298	92 304	92 309	92 314	92 319
838	92 324	92 330	92 335	92 340	92 345	92 350	92 355	92 361	92 366	92 371
839	92 376	92 381	92 387	92 392	92 397	92 402	92 407	92 412	92 418	92 423
840	92 428	92 433	92 438	92 443	92 449	92 454	92 459	92 464	92 469	92 474
841	92 480	92 485	92 490	92 495	92 500	92 505	92 511	92 516	92 521	92 526
842	92 531	92 536	92 542	92 547	92 552	92 557	92 562	92 567	92 572	92 578
843	92 583	92 588	92 593	92 598	92 603	92 609	92 614	92 619	92 624	92 629
844	92 634	92 639	92 645	92 650	92 655	92 660	92 665	92 670	92 675	92 681
845	92 686	92 691	92 696	92 701	92 706	92 711	92 716	92 722	92 727	92 732
846	92 737	92 742	92 747	92 752	92 758	92 763	92 768	92 773	92 778	92 783
847	92 788	92 793	92 799	92 804	92 809	92 814	92 819	92 824	92 829	92 834
848	92 840	92 845	92 850	92 855	92 860	92 865	92 870	92 875	92 881	92 886
849	92 891	92 896	92 901	92 906	92 911	92 916	92 921	92 927	92 932	92 937
850	92 942	92 947	92 952	92 957	92 962	92 967	92 973	92 978	92 983	92 988
851	92 993	92 998	93 003	93 008	93 013	93 018	93 024	93 029	93 034	93 039
852	93 044	93 049	93 054	93 059	93 064	93 069	93 075	93 080	93 085	93 090
853	93 095	93 100	93 105	93 110	93 115	93 120	93 125	93 131	93 136	93 141
854	93 146	93 151	93 156	93 161	93 166	93 171	93 176	93 181	93 186	93 192

No.	0	1	2	3	4	5	6	7	8	9
855	93 197	93 202	93 207	93 212	93 217	93 222	93 227	93 232	93 237	93 242
856	93 247	93 252	93 258	93 263	93 268	93 273	93 278	93 283	93 288	93 293
857	93 298	93 303	93 308	93 313	93 318	93 323	93 328	93 334	93 339	93 344
858	93 349	93 354	93 359	93 364	93 369	93 374	93 379	93 384	93 389	93 394
859	93 399	93 404	93 409	93 414	93 420	93 425	93 430	93 435	93 440	93 445
860	93 450	93 455	93 460	93 465	93 470	93 475	93 480	93 485	93 490	93 495
861	93 500	93 505	93 510	93 515	93 520	93 526	93 531	93 536	93 541	93 546
862	93 551	93 556	93 561	93 566	93 571	93 576	93 581	93 586	93 591	93 596
863	93 601	93 606	93 611	93 616	93 621	93 626	93 631	93 636	93 641	93 646
864	93 651	93 656	93 661	93 666	93 671	93 676	93 682	93 687	93 692	93 697
865	93 702	93 707	93 712	93 717	93 722	93 727	93 732	93 737	93 742	93 747
866	93 752	93 757	93 762	93 767	93 772	93 777	93 782	93 787	93 792	93 797
867	93 802	93 807	93 812	93 817	93 822	93 827	93 832	93 837	93 842	93 847
868	93 852	93 857	93 862	93 867	93 872	93 877	93 882	93 887	93 892	93 897
869	93 902	93 907	93 912	93 917	93 922	93 927	93 932	93 937	93 942	93 947
870	93 952	93 957	93 962	93 967	93 972	93 977	93 982	93 987	93 992	93 997
871	94 002	94 007	94 012	94 017	94 022	94 027	94 032	94 037	94 042	94 047
872	94 052	94 057	94 062	94 067	94 072	94 077	94 082	94 086	94 091	94 096
873	94 101	94 106	94 111	94 116	94 121	94 126	94 131	94 136	94 141	94 146
874	94 151	94 156	94 161	94 166	94 171	94 176	94 181	94 186	94 191	94 196
875	94 201	94 206	94 211	94 216	94 221	94 226	94 231	94 236	94 240	94 245
876	94 250	94 255	94 260	94 265	94 270	94 275	94 280	94 285	94 290	94 295
877	94 300	94 305	94 310	94 315	94 320	94 325	94 330	94 335	94 340	94 345
878	94 349	94 354	94 359	94 364	94 369	94 374	94 379	94 384	94 389	94 394
879	94 399	94 404	94 409	94 414	94 419	94 424	94 429	94 433	94 438	94 443
880	94 448	94 453	94 458	94 463	94 468	94 473	94 478	94 483	94 488	94 493
881	94 498	94 503	94 507	94 512	94 517	94 522	94 527	94 532	94 537	94 542
882	94 547	94 552	94 557	94 562	94 567	94 571	94 576	94 581	94 586	94 591
883	94 596	94 601	94 606	94 611	94 616	94 621	94 626	94 630	94 635	94 640
884	94 645	94 650	94 655	94 660	94 665	94 670	94 675	94 680	94 685	94 689
885	94 694	94 699	94 704	94 709	94 714	94 719	94 724	94 729	94 734	94 738
886	94 743	94 748	94 753	94 758	94 763	94 768	94 773	94 778	94 783	94 787
887	94 792	94 797	94 802	94 807	94 812	94 817	94 822	94 827	94 832	94 836
888	94 841	94 846	94 851	94 856	94 861	94 866	94 871	94 876	94 880	94 885
889	94 890	94 895	94 900	94 905	94 910	94 915	94 919	94 924	94 929	94 934
890	94 939	94 944	94 949	94 954	94 959	94 963	94 968	94 973	94 978	94 983
891	94 988	94 993	94 998	95 002	95 007	95 012	95 017	95 022	95 027	95 032
892	95 036	95 041	95 046	95 051	95 056	95 061	95 066	95 071	95 075	95 080
893	95 085	95 090	95 095	95 100	95 105	95 109	95 114	95 119	95 124	95 129
894	95 134	95 139	95 143	95 148	95 153	95 158	95 163	95 168	95 173	95 177
895	95 182	95 187	95 192	95 197	95 202	95 207	95 211	95 216	95 221	95 226
896	95 231	95 236	95 240	95 245	95 250	95 255	95 260	95 265	95 270	95 274
897	95 279	95 284	95 289	95 294	95 299	95 303	95 308	95 313	95 318	95 323
898	95 328	95 332	95 337	95 342	95 347	95 352	95 357	95 361	95 366	95 371
899	95 376	95 381	95 386	95 390	95 395	95 400	95 405	95 410	95 415	95 419

No.	0	1	2	3	4	5	6	7	8	9
900	95 424	95 429	95 434	95 439	95 444	95 448	95 453	95 458	95 463	95 468
901	95 472	95 477	95 482	95 487	95 492	95 497	95 501	95 506	95 511	95 516
902	95 521	95 525	95 530	95 535	95 540	95 545	95 550	95 554	95 559	95 564
903	95 569	95 574	95 578	95 583	95 588	95 593	95 598	95 602	95 607	95 612
904	95 617	95 622	95 626	95 631	95 636	95 641	95 646	95 650	95 655	95 660
905	95 665	95 670	95 674	95 679	95 684	95 689	95 694	95 698	95 703	95 708
906	95 713	95 718	95 722	95 727	95 732	95 737	95 742	95 746	95 751	95 756
907	95 761	95 766	95 770	95 775	95 780	95 785	95 789	95 794	95 799	95 804
908	95 809	95 813	95 818	95 823	95 828	95 832	95 837	95 842	95 847	95 852
909	95 856	95 861	95 866	95 871	95 875	95 880	95 885	95 890	95 895	95 899
910	95 904	95 909	95 914	95 918	95 923	95 928	95 933	95 938	95 942	95 947
911	95 952	95 957	95 961	95 966	95 971	95 976	95 980	95 985	95 990	95 995
912	95 999	96 004	96 009	96 014	96 019	96 023	96 028	96 033	96 038	96 042
913	96 047	96 052	96 057	96 061	96 066	96 071	96 076	96 080	96 085	96 090
914	96 095	96 099	96 104	96 109	96 114	96 118	96 123	96 128	96 133	96 137
915	96 142	96 147	96 152	96 156	96 161	96 166	96 171	96 175	96 180	96 185
916	96 190	96 194	96 199	96 204	96 209	96 213	96 218	96 223	96 227	96 232
917	96 237	96 242	96 246	96 251	96 256	96 261	96 265	96 270	96 275	96 280
918	96 284	96 289	96 294	96 298	96 303	96 308	96 313	96 317	96 322	96 327
919	96 332	96 336	96 341	96 346	96 350	96 355	96 360	96 365	96 369	96 374
920	96 379	96 384	96 388	96 393	96 398	96 402	96 407	96 412	96 417	96 421
921	96 426	96 431	96 435	96 440	96 445	96 450	96 454	96 459	96 464	96 468
922	96 473	96 478	96 483	96 487	96 492	96 497	96 501	96 506	96 511	96 515
923	96 520	96 525	96 530	96 534	96 539	96 544	96 548	96 553	96 558	96 562
924	96 567	96 572	96 577	96 581	96 586	96 591	96 595	96 600	96 605	96 609
925	96 614	96 619	96 624	96 628	96 633	96 638	96 642	96 647	96 652	96 656
926	96 661	96 666	96 670	96 675	96 680	96 685	96 689	96 694	96 699	96 703
927	96 708	96 713	96 717	96 722	96 727	96 731	96 736	96 741	96 745	96 750
928	96 755	96 759	96 764	96 769	96 774	96 778	96 783	96 788	96 792	96 797
929	96 802	96 806	96 811	96 816	96 820	96 825	96 830	96 834	96 839	96 844
930	96 848	96 853	96 858	96 862	96 867	96 872	96 876	96 881	96 886	96 890
931	96 895	96 900	96 904	96 909	96 914	96 918	96 923	96 928	96 932	96 937
932	96 942	96 946	96 951	96 956	96 960	96 965	96 970	96 974	96 979	96 984
933	96 988	96 993	96 997	97 002	97 007	97 011	97 016	97 021	97 025	97 030
934	97 035	97 039	97 044	97 049	97 053	97 058	97 063	97 067	97 072	97 077
935	97 081	97 086	97 090	97 095	97 100	97 104	97 109	97 114	97 118	97 123
936	97 128	97 132	97 137	97 142	97 146	97 151	97 155	97 160	97 165	97 169
937	97 174	97 179	97 183	97 188	97 192	97 197	97 202	97 206	97 211	97 216
938	97 220	97 225	97 230	97 234	97 239	97 243	97 248	97 253	97 257	97 262
939	97 267	97 271	97 276	97 280	97 285	97 290	97 294	97 299	97 304	97 308
940	97 313	97 317	97 322	97 327	97 331	97 336	97 340	97 345	97 350	97 354
941	97 359	97 364	97 368	97 373	97 377	97 382	97 387	97 391	97 396	97 400
942	97 405	97 410	97 414	97 419	97 424	97 428	97 433	97 437	97 442	97 447
943	97 451	97 456	97 460	97 465	97 470	97 474	97 479	97 483	97 488	97 493
944	97 497	97 502	97 506	97 511	97 516	97 520	97 525	97 529	97 534	97 539

No.	0	1	2	3	4	5	6	7	8	9
945	97 543	97 548	97 552	97 557	97 562	97 566	97 571	97 575	97 580	97 585
946	97 589	97 594	97 598	97 603	97 607	97 612	97 617	97 621	97 626	97 630
947	97 635	97 640	97 644	97 649	97 653	97 658	97 663	97 667	97 672	97 676
948	97 681	97 685	97 690	97 695	97 699	97 704	97 708	97 713	97 717	97 722
949	97 727	97 731	97 736	97 740	97 745	97 749	97 754	97 759	97 763	97 768
950	97 772	97 777	97 782	97 786	97 791	97 795	97 800	97 804	97 809	97 813
951	97 818	97 823	97 827	97 832	97 836	97 841	97 845	97 850	97 855	97 859
952	97 864	97 868	97 873	97 877	97 882	97 886	97 891	97 896	97 900	97 905
953	97 909	97 914	97 918	97 923	97 928	97 932	97 937	97 941	97 946	97 950
954	97 955	97 959	97 964	97 968	97 973	97 978	97 982	97 987	97 991	97 996
955	98 000	98 005	98 009	98 014	98 019	98 023	98 028	98 032	98 037	98 041
956	98 046	98 050	98 055	98 059	98 064	98 068	98 073	98 078	98 082	98 087
957	98 091	98 096	98 100	98 105	98 109	98 114	98 118	98 123	98 127	98 132
958	98 137	98 141	98 146	98 150	98 155	98 159	98 164	98 168	98 173	98 177
959	98 182	98 186	98 191	98 195	98 200	98 204	98 209	98 214	98 218	98 223
960	98 227	98 232	98 236	98 241	98 245	98 250	98 254	98 259	98 263	98 268
961	98 272	98 277	98 281	98 286	98 290	98 295	98 299	98 304	98 308	98 313
962	98 318	98 322	98 327	98 331	98 336	98 340	98 345	98 349	98 354	98 358
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964	98 408	98 412	98 417	98 421	98 426	98 430	98 435	98 439	98 444	98 448
965	98 453	98 457	98 462	98 466	98 471	98 475	98 480	98 484	98 489	98 493
966	98 498	98 502	98 507	98 511	98 516	98 520	98 525	98 529	98 534	98 538
967	98 543	98 547	98 552	98 556	98 561	98 565	98 570	98 574	98 579	98 583
968	98 588	98 592	98 597	98 601	98 605	98 610	98 614	98 619	98 623	98 628
969	98 632	98 637	98 641	98 646	98 650	98 655	98 659	98 664	98 668	98 673
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971	98 722	98 726	98 731	98 735	98 740	98 744	98 749	98 753	98 758	98 762
972	98 767	98 771	98 776	98 780	98 784	98 789	98 793	98 798	98 802	98 807
973	98 811	98 816	98 820	98 825	98 829	98 834	98 838	98 843	98 847	98 851
974	98 856	98 860	98 865	98 869	98 874	98 878	98 883	98 887	98 892	98 896
975	98 900	98 905	98 909	98 914	98 918	98 923	98 927	98 932	98 936	98 941
976	98 945	98 949	98 954	98 958	98 963	98 967	98 972	98 976	98 981	98 985
977	98 989	98 994	98 998	99 003	99 007	99 012	99 016	99 021	99 025	99 029
978	99 034	99 038	99 043	99 047	99 052	99 056	99 061	99 065	99 069	99 074
979	99 078	99 083	99 087	99 092	99 096	99 100	99 105	99 109	99 114	99 118
980	99 123	99 127	99 131	99 136	99 140	99 145	99 149	99 154	99 158	99 162
981	99 167	99 171	99 176	99 180	99 185	99 189	99 193	99 198	99 202	99 207
982	99 211	99 216	99 220	99 224	99 229	99 233	99 238	99 242	99 247	99 251
983	99 255	99 260	99 264	99 269	99 273	99 277	99 282	99 286	99 291	99 295
984	99 300	99 304	99 308	99 313	99 317	99 322	99 326	99 330	99 335	99 339
985	99 344	99 348	99 352	99 357	99 361	99 366	99 370	99 374	99 379	99 383
986	99 388	99 392	99 396	99 401	99 405	99 410	99 414	99 419	99 423	99 427
987	99 432	99 436	99 441	99 445	99 449	99 454	99 458	99 463	99 467	99 471
988	99 476	99 480	99 484	99 489	99 493	99 498	99 502	99 506	99 511	99 515
989	99 520	99 524	99 528	99 533	99 537	99 542	99 546	99 550	99 555	99 559

No.	0	1	2	3	4	5	6	7	8	9
990	99 564	99 568	99 572	99 577	99 581	99 585	99 590	99 594	99 599	99 603
991	99 607	99 612	99 616	99 621	99 625	99 629	99 634	99 638	99 642	99 647
992	99 651	99 656	99 660	99 664	99 669	99 673	99 677	99 682	99 686	99 691
993	99 695	99 699	99 704	99 708	99 712	99 717	99 721	99 726	99 730	99 734
994	99 739	99 743	99 747	99 752	99 756	99 760	99 765	99 769	99 774	99 778
995	99 782	99 787	99 791	99 795	99 800	99 804	99 808	99 813	99 817	99 822
996	99 826	99 830	99 835	99 839	99 843	99 848	99 852	99 856	99 861	99 865
997	99 870	99 874	99 878	99 883	99 887	99 891	99 896	99 900	99 904	99 909
998	99 913	99 917	99 922	99 926	99 930	99 935	99 939	99 944	99 948	99 952
999	99 957	99 961	99 965	99 970	99 974	99 978	99 983	99 987	99 991	99 996
1000	00 000	00 004	00 009	00 013	00 017	00 022	00 026	00 030	00 035	00 039

APPENDIX C

COMPOUND INTEREST AND OTHER COMPUTATIONS[1]

TABLE I—PART 1

AMOUNT OF $1 AT COMPOUND INTEREST

PERIODS	1%	1 1/4%	1 1/2%	1 3/4%	2%	2 1/4%	2 1/2%
0	1.	1.	1.	1.	1.	1.	1.
1	1.01	1.0125	1.015	1.0175	1.02	1.0225	1.025
2	1.0201	1.02515625	1.030225	1.03530625	1.0404	1.04550625	1.050625
3	1.030301	1.03797070	1.04567838	1.05342411	1.061208	1.06903014	1.07689063
4	1.04060401	1.05094534	1.06136355	1.07185903	1.08243216	1.09308332	1.10381289
5	1.05101005	1.06408215	1.07728400	1.09061656	1.10408080	1.11767769	1.13140821
6	1.06152015	1.07738318	1.09344326	1.10970235	1.12616242	1.14282544	1.15969342
7	1.07213535	1.09085047	1.10984491	1.12912215	1.14868567	1.16853901	1.18868575
8	1.08285671	1.10448610	1.12649259	1.14888178	1.17165938	1.19483114	1.21840290
9	1.09368527	1.11829218	1.14338998	1.16898721	1.19509257	1.22171484	1.24886297
10	1.10462213	1.13227083	1.16054083	1.18944449	1.21899442	1.24920343	1.28008454
11	1.11566835	1.14642422	1.17794894	1.21025977	1.24337431	1.27731050	1.31208666
12	1.12682503	1.16075452	1.19561817	1.23143931	1.26824179	1.30604999	1.34488882
13	1.13809328	1.17526395	1.21355244	1.25298950	1.29360663	1.33543611	1.37851104
14	1.14947421	1.18995475	1.23175573	1.27491682	1.31947876	1.36548343	1.41297382
15	1.16096896	1.20482918	1.25023207	1.29722786	1.34586834	1.39620680	1.44829817
16	1.17257864	1.21988955	1.26898555	1.31992935	1.37278571	1.42762146	1.48450562
17	1.18430443	1.23513817	1.28802033	1.34302811	1.40024142	1.45974294	1.52161826
18	1.19614748	1.25057739	1.30734064	1.36653111	1.42824625	1.49258716	1.55965872
19	1.20810895	1.26620961	1.32695075	1.39044540	1.45681117	1.52617037	1.59865019
20	1.22019004	1.28203723	1.34685501	1.41477820	1.48594740	1.56050920	1.63861644
21	1.23239194	1.29806270	1.36705783	1.43953681	1.51566634	1.59562066	1.67958185
22	1.24471586	1.31428848	1.38756370	1.46472871	1.54597967	1.63152212	1.72157140
23	1.25716302	1.33071709	1.40837715	1.49036146	1.57689926	1.66823137	1.76461068
24	1.26973465	1.34735105	1.42950281	1.51644279	1.60843725	1.70576658	1.80872595

25	1.28243200	1.36419294	1.45094535	1.54298054	1.64060599	1.74414632	1.85394410
26	1.29525631	1.38124535	1.47270953	1.56998269	1.67341811	1.78338962	1.90029270
27	1.30820888	1.39851092	1.49480018	1.59745739	1.70688648	1.82351588	1.94780002
28	1.32129097	1.41599230	1.51722218	1.62541290	1.74102421	1.86454499	1.99649502
29	1.33450388	1.43369221	1.53998051	1.65385762	1.77584469	1.90649725	2.04640739
30	1.34784892	1.45161336	1.56308022	1.68280013	1.81136158	1.94939344	2.09756758
31	1.36132740	1.46975853	1.58652642	1.71224913	1.84758882	1.99325479	2.15000677
32	1.37494068	1.48813051	1.61032432	1.74221349	1.88454059	2.03810303	2.20375694
33	1.38869009	1.50673214	1.63447918	1.77270223	1.92223140	2.08396034	2.25885086
34	1.40257699	1.52556629	1.65899637	1.80372452	1.96067603	2.13084945	2.31532213
35	1.41660276	1.54463587	1.68388132	1.83528970	1.99988955	2.17879356	2.37320519
36	1.43076878	1.56394382	1.70913954	1.86740727	2.03988734	2.22781642	2.43253532
37	1.44507647	1.58349312	1.73477663	1.90008689	2.08068509	2.27794229	2.49334870
38	1.45952724	1.60328678	1.76079828	1.93333841	2.12229879	2.32919599	2.55568242
39	1.47412251	1.62332787	1.78721025	1.96717184	2.16474477	2.38160290	2.61957448
40	1.48886373	1.64361946	1.81401841	2.00159734	2.20803966	2.43518897	2.68506384
41	1.50375237	1.66416471	1.84122868	2.03662530	2.25220046	2.48998072	2.75219043
42	1.51878989	1.68496677	1.86884712	2.07226624	2.29724447	2.54600528	2.82099520
43	1.53397779	1.70602885	1.89687982	2.10853090	2.34318936	2.60329040	2.89152008
44	1.54931757	1.72735421	1.92533302	2.14543019	2.39005314	2.66186444	2.96380808
45	1.56481075	1.74894614	1.95421301	2.18297522	2.43785421	2.72175639	3.03790328
46	1.58045885	1.77080797	1.98352621	2.22117728	2.48661129	2.78299590	3.11385086
47	1.59626344	1.79294306	2.01327910	2.26004789	2.53634351	2.84561331	3.19169713
48	1.61222608	1.81535485	2.04347829	2.29959872	2.58707039	2.90963961	3.27148956
49	1.62834834	1.83804679	2.07413046	2.33984170	2.63881179	2.97510650	3.35327680
50	1.64463182	1.86102237	2.10524242	2.38078893	2.69158803	3.04204640	3.43710872
55	1.72852457	1.98028070	2.26794398	2.59652785	2.97173067	3.40002740	3.88877303
60	1.81669670	2.10718135	2.44321978	2.83181628	3.28103079	3.80013479	4.39978975
65	1.90936649	2.24221407	2.63204158	3.08842574	3.62252311	4.24732588	4.97795826
70	2.00676337	2.38589997	2.83545629	3.36828827	3.99955822	4.74714140	5.63210286
75	2.10912847	2.53879358	3.05459171	3.67351098	4.41583546	5.30577405	6.37220743
80	2.21671522	2.70148494	3.29066279	4.00639192	4.87543916	5.93014530	7.20956782
85	2.32978997	2.87460191	3.54497838	4.36943740	5.38287878	6.62799112	8.15696424
90	2.44863267	3.05881260	3.81894851	4.76538080	5.94313313	7.40795782	9.22885633
95	2.57353755	3.25482789	4.11409214	5.19720324	6.56169920	8.27970921	10.44160385
100	2.70481383	3.46340427	4.43204565	5.66815594	7.24464612	9.25404630	11.81371635

[1] C. E. Sprague and L. L. Perrine, The Accountancy of Investment, 1914, pages 336–351.

TABLE I—PART 2

AMOUNT OF $1 AT COMPOUND INTEREST

PERIODS	2 3/4%	3%	3 1/2%	4%	4 1/2%	5%	6%
0	1.	1.	1.	1.	1.	1.	1.
1	1.0275	1.03	1.035	1.04	1.045	1.05	1.06
2	1.05575625	1.0609	1.071225	1.0816	1.092025	1.1025	1.1236
3	1.08478955	1.092727	1.10871788	1.124864	1.14116613	1.157625	1.191016
4	1.11462126	1.12550881	1.14752300	1.16985856	1.19251860	1.21550625	1.26247696
5	1.14527334	1.15927407	1.18768631	1.21665290	1.24618194	1.27628156	1.33822558
6	1.17676836	1.19405230	1.22925533	1.26531902	1.30226012	1.34009564	1.41851911
7	1.20912949	1.22987387	1.27227926	1.31593178	1.36086183	1.40710042	1.50363026
8	1.24238055	1.26677008	1.31680904	1.36856905	1.42210061	1.47745544	1.59384807
9	1.27654602	1.30477318	1.36289735	1.42331181	1.48609514	1.55132822	1.68947896
10	1.31165103	1.34391638	1.41059876	1.48024428	1.55296942	1.62889463	1.79084770
11	1.34772144	1.38423387	1.45996972	1.53945406	1.62285305	1.71033936	1.89829856
12	1.38478378	1.42576089	1.51106866	1.60103222	1.69588143	1.79585633	2.01219647
13	1.42286533	1.46853371	1.56395606	1.66507351	1.77219610	1.88564914	2.13292826
14	1.46199413	1.51258972	1.61869452	1.73167645	1.85194492	1.97993160	2.26090396
15	1.50219896	1.55796742	1.67534883	1.80094351	1.93528244	2.07892818	2.39655819
16	1.54350944	1.60470644	1.73398604	1.87298125	2.02237015	2.18287459	2.54035168
17	1.58595595	1.65284763	1.79467555	1.94790050	2.11337681	2.29201832	2.69277279
18	1.62956973	1.70243306	1.85748920	2.02581652	2.20847877	2.40661923	2.85433915
19	1.67438290	1.75350605	1.92250132	2.10684918	2.30786031	2.52695020	3.02559950
20	1.72042843	1.80611123	1.98978886	2.19112314	2.41171402	2.65329771	3.20713547
21	1.76774021	1.86029457	2.05943147	2.27876807	2.52024116	2.78596259	3.39956360
22	1.81635307	1.91610341	2.13151158	2.36991879	2.63365201	2.92526072	3.60353742
23	1.86630278	1.97358651	2.20611448	2.46471554	2.75216635	3.07152376	3.81974966
24	1.91762610	2.03279411	2.28322849	2.56330416	2.87601383	3.22509994	4.04893464
25	1.97036082	2.09377793	2.36324498	2.66583633	3.00543446	3.38635494	4.29187072
26	2.02454575	2.15659127	2.44595856	2.77246978	3.14067901	3.55567269	4.54938296
27	2.08022075	2.22128901	2.53156711	2.88336858	3.28200956	3.73345632	4.82234594
28	2.13742682	2.28792768	2.62017196	2.99870332	3.42969999	3.92012914	5.11168670
29	2.19620606	2.35656551	2.71187798	3.11865145	3.58403649	4.11613560	5.41838790

30	2.25660173	2.42726247	2.80679370	3.24339751	3.74531813	4.32194238	5.74349117
31	2.31865828	2.50008035	2.90503148	3.37313341	3.91385745	4.53803949	6.08810064
32	2.38242138	2.57508276	3.00670759	3.50805875	4.08998104	4.76494147	6.45338668
33	2.44793797	2.65233524	3.11194235	3.64838110	4.27403018	5.00318854	6.84058988
34	2.51525626	2.73190530	3.22086033	3.79431634	4.46636154	5.25334797	7.25102528
35	2.58442581	2.81386245	3.33359045	3.94608899	4.66734781	5.51601537	7.68608679
36	2.65549752	2.89827833	3.45026611	4.10393255	4.87737846	5.79181614	8.14725200
37	2.72852370	2.98522668	3.57102543	4.26808986	5.09686049	6.08140694	8.63608712
38	2.80355810	3.07478348	3.69601132	4.43881345	5.32621921	6.38547729	9.15425235
39	2.88065595	3.16702698	3.82537171	4.61636599	5.56589908	6.70475115	9.70350749
40	2.95987399	3.26203779	3.95925972	4.80102063	5.81636454	7.03998871	10.28571794
41	3.04127052	3.35989893	4.09783381	4.99306145	6.07810094	7.39198815	10.90286101
42	3.12490546	3.46069589	4.24125799	5.19278391	6.35161548	7.76158756	11.55703267
43	3.21084036	3.56451677	4.38970202	5.40049527	6.63743818	8.14966693	12.25045463
44	3.29913847	3.67145227	4.54334160	5.61651508	6.93612290	8.55715028	12.98548191
45	3.38986478	3.78159584	4.70235855	5.84117568	7.24824843	8.98500779	13.76461083
46	3.48308606	3.89504372	4.86694110	6.07482271	7.57441961	9.43425818	14.59048748
47	3.57887093	4.01189503	5.03728404	6.31781562	7.91526849	9.90597109	15.46591673
48	3.67728988	4.13225188	5.21358898	6.57052824	8.27145557	10.40126965	16.39387173
49	3.77841535	4.25621944	5.39606459	6.83334937	8.64367107	10.92133313	17.37750403
50	3.88232177	4.38390602	5.58492686	7.10668335	9.03263627	11.46739979	18.42015427
55	4.44631964	5.08214859	6.63314114	8.64636692	11.25630817	14.63563092	24.65032159
60	5.09225136	5.89160310	7.87809090	10.51962741	14.02740793	18.67918589	32.98769085
65	5.83201974	6.82998273	9.35670068	12.79873522	17.48070239	23.83990056	44.14497165
70	6.67925676	7.91782191	11.11282526	15.57161835	21.78413558	30.42642554	59.07593018
75	7.64957472	9.17892567	13.19855038	18.94525466	27.14699629	38.83268592	79.05692079
80	8.76085402	10.64089056	15.67573754	23.04979907	33.83009643	49.56144107	105.79599348
85	10.03357258	12.33570855	18.61785881	28.04360494	42.15845513	63.25435344	141.57890449
90	11.49118322	14.30046711	22.11217595	34.11933334	52.53710530	80.73036505	189.46451123
95	13.16054584	16.57816077	26.26232856	41.51138594	65.47079168	103.03467645	253.54625498
100	15.07242234	19.21863198	31.19140798	50.50494818	81.58851803	131.50125785	339.30208351

Table II—Part 1

Present Worth of $1 at Compound Interest

Periods	1%	1 1/4%	1 1/2%	1 3/4%	2%	2 1/4%	2 1/2%
0	1.	1.	1.	1.	1.	1.	1.
1	0.99009901	0.98765432	0.98522167	0.98280098	0.98039216	0.97799511	0.97560976
2	0.98029605	0.97546106	0.97066175	0.96589777	0.96116878	0.95647444	0.95181440
3	0.97059015	0.96341833	0.95631699	0.94928528	0.94232233	0.93542732	0.92859941
4	0.96098034	0.95152428	0.94218423	0.93295851	0.92384543	0.91484335	0.90595064
5	0.95146569	0.93977706	0.92826033	0.91691254	0.90573081	0.89471232	0.88385429
6	0.94204524	0.92817488	0.91454219	0.90114254	0.88797138	0.87502427	0.86229687
7	0.93271805	0.91671593	0.90102679	0.88564378	0.87056018	0.85576946	0.84126524
8	0.92348322	0.90539845	0.88771112	0.87041157	0.85349037	0.83693835	0.82074657
9	0.91433982	0.89422069	0.87459224	0.85544135	0.83675527	0.81852161	0.80072836
10	0.90528695	0.88318093	0.86166723	0.84072860	0.82034830	0.80051013	0.78119840
11	0.89632372	0.87227746	0.84893323	0.82626889	0.80426304	0.78289499	0.76214478
12	0.88744923	0.86150860	0.83638742	0.81205788	0.78849318	0.76566748	0.74355589
13	0.87866260	0.85087269	0.82402702	0.79809128	0.77303253	0.74881905	0.72542038
14	0.86996297	0.84036809	0.81184928	0.78436490	0.75787502	0.73234137	0.70772720
15	0.86134947	0.82999318	0.79985150	0.77087459	0.74301473	0.71622628	0.69046556
16	0.85282126	0.81974635	0.78803104	0.75761631	0.72844581	0.70046580	0.67362493
17	0.84437749	0.80962602	0.77638526	0.74458605	0.71416256	0.68505212	0.65719506
18	0.83601731	0.79963064	0.76491159	0.73177990	0.70015937	0.66997763	0.64116591
19	0.82773992	0.78975866	0.75360747	0.71919401	0.68643076	0.65523484	0.62552772
20	0.81954447	0.78000855	0.74247042	0.70682458	0.67297133	0.64081647	0.61027094
21	0.81143017	0.77037881	0.73149795	0.69466789	0.65977582	0.62671538	0.59538629
22	0.80339621	0.76086796	0.72068763	0.68272028	0.64683904	0.61292457	0.58086467
23	0.79544179	0.75147453	0.71003708	0.67097817	0.63415592	0.59943724	0.56669724
24	0.78756613	0.74219707	0.69954392	0.65943800	0.62172149	0.58624668	0.55287535
25	0.77976844	0.73303414	0.68920583	0.64809632	0.60953087	0.57334639	0.53939059
26	0.77204796	0.72398434	0.67902052	0.63694970	0.59757928	0.56072997	0.52623472
27	0.76440392	0.71504626	0.66898574	0.62599479	0.58586204	0.54839117	0.51339973
28	0.75683557	0.70621853	0.65909925	0.61522829	0.57437455	0.53632388	0.50087778
29	0.74934215	0.69749978	0.64935887	0.60464697	0.56311231	0.52452213	0.48866125

30	0.74192292	0.68888867	0.63976243	0.59424764	0.55207089	0.51298008	0.47674269
31	0.73457715	0.68038387	0.63030781	0.58402716	0.54124597	0.50169201	0.46511481
32	0.72730411	0.67198407	0.62099292	0.57398247	0.53063330	0.49065233	0.45377055
33	0.72010307	0.66368797	0.61181568	0.56411053	0.52022873	0.47985558	0.44270298
34	0.71297334	0.65549429	0.60277407	0.55440839	0.51002817	0.46929641	0.43190534
35	0.70591420	0.64740177	0.59386608	0.54487311	0.50002761	0.45896960	0.42137107
36	0.69892495	0.63940916	0.58508974	0.53550183	0.49022315	0.44887002	0.41109372
37	0.69200490	0.63151522	0.57644309	0.52629172	0.48061093	0.43899268	0.40106705
38	0.68515337	0.62371873	0.56792423	0.51724002	0.47118719	0.42933270	0.39128492
39	0.67836967	0.61601850	0.55953126	0.50834400	0.46194822	0.41988528	0.38174139
40	0.67165314	0.60841334	0.55126232	0.49960098	0.45289042	0.41064575	0.37243062
41	0.66500311	0.60090206	0.54311559	0.49100834	0.44401021	0.40160954	0.36334695
42	0.65841892	0.59348352	0.53508925	0.48256348	0.43530413	0.39277216	0.35448483
43	0.65189992	0.58615656	0.52718153	0.47426386	0.42676875	0.38412925	0.34583886
44	0.64544546	0.57892006	0.51939067	0.46610699	0.41840074	0.37567653	0.33740376
45	0.63905492	0.57177290	0.51171494	0.45809040	0.41019680	0.36740981	0.32917440
46	0.63272764	0.56471397	0.50415265	0.45021170	0.40215373	0.35932500	0.32114576
47	0.62646301	0.55774219	0.49670212	0.44246850	0.39426836	0.35141809	0.31331294
48	0.62026041	0.55085649	0.48936170	0.43485848	0.38653761	0.34368518	0.30567116
49	0.61411921	0.54405579	0.48212975	0.42737934	0.37895844	0.33612242	0.29821576
50	0.60803882	0.53733905	0.47500468	0.42002883	0.37152788	0.32872608	0.29094221
55	0.57852808	0.50497892	0.44092800	0.38512970	0.33650425	0.29411528	0.25715052
60	0.55044962	0.47456760	0.40929597	0.35313025	0.30478227	0.26314856	0.22728359
65	0.52373392	0.44598775	0.37993321	0.32378956	0.27605069	0.23544226	0.20088557
70	0.49831486	0.41912905	0.35267692	0.29688670	0.25002761	0.21065309	0.17755358
75	0.47412949	0.39388787	0.32737599	0.27221914	0.22645771	0.18847391	0.15693149
80	0.45111794	0.37016679	0.30389015	0.24960114	0.20510973	0.16862993	0.13870457
85	0.42922324	0.34787426	0.28208917	0.22886242	0.18577420	0.15087528	0.12259463
90	0.40839119	0.32692425	0.26185218	0.20984682	0.16826142	0.13498997	0.10835579
95	0.38857020	0.30723591	0.24306699	0.19241118	0.15239955	0.12077719	0.09577073
100	0.36971121	0.28873326	0.22562944	0.17642422	0.13803297	0.10806084	0.08464737

TABLE II—PART 2

PRESENT WORTH OF $1 AT COMPOUND INTEREST

PERIODS	2 3/4%	3%	3 1/2%	4%	4 1/2%	5%	6%
0	1.	1.	1.	1.	1.	1.	1.
1	0.97323601	0.97087379	0.96618357	0.96153846	0.95693780	0.95238095	0.94339623
2	0.94718833	0.94259591	0.93351070	0.92455621	0.91572995	0.90702948	0.88999644
3	0.92183779	0.91514166	0.90194271	0.88899636	0.87629660	0.86383760	0.83961928
4	0.89716573	0.88848705	0.87144223	0.85480419	0.83856134	0.82270247	0.79209366
5	0.87315400	0.86260878	0.84197317	0.82192711	0.80245105	0.78352617	0.74725817
6	0.84978491	0.83748426	0.81350064	0.79031453	0.76789574	0.74621540	0.70496054
7	0.82704128	0.81309151	0.78599096	0.75991781	0.73482846	0.71068133	0.66505711
8	0.80490635	0.78940923	0.75941156	0.73069021	0.70318513	0.67683936	0.62741237
9	0.78336385	0.76641673	0.73373097	0.70258674	0.67290443	0.64460892	0.59189846
10	0.76239791	0.74409391	0.70891881	0.67556417	0.64392768	0.61391325	0.55839478
11	0.74199310	0.72242128	0.68494571	0.64958093	0.61619874	0.58467929	0.52678753
12	0.72213440	0.70137988	0.66178330	0.62459705	0.58966386	0.55683742	0.49696936
13	0.70280720	0.68095134	0.63940415	0.60057409	0.56427164	0.53032135	0.46883902
14	0.68399728	0.66111781	0.61778179	0.57747508	0.53997286	0.50506795	0.44230096
15	0.66569078	0.64186195	0.59689062	0.55526450	0.51672044	0.48101710	0.41726506
16	0.64787424	0.62316694	0.57670591	0.53390818	0.49446932	0.45811152	0.39364628
17	0.63053454	0.60501645	0.55720378	0.51337325	0.47317639	0.43629669	0.37136442
18	0.61365892	0.58739461	0.53836114	0.49362812	0.45280037	0.41552065	0.35034379
19	0.59723496	0.57028603	0.52015569	0.47464242	0.43330179	0.39573396	0.33051301
20	0.58125057	0.55367575	0.50256588	0.45638695	0.41464286	0.37688948	0.31180473
21	0.56569398	0.53754928	0.48557090	0.43883360	0.39678743	0.35894236	0.29415540
22	0.55055375	0.52189250	0.46915063	0.42195539	0.37970089	0.34184987	0.27750510
23	0.53581874	0.50669175	0.45328563	0.40572633	0.36335013	0.32557131	0.26179726
24	0.52147809	0.49193374	0.43795713	0.39012147	0.34770347	0.31006791	0.24697855
25	0.50752126	0.47760557	0.42314699	0.37511680	0.33273060	0.29530277	0.23299863
26	0.49393796	0.46369473	0.40883767	0.36068923	0.31840248	0.28124073	0.21981003
27	0.48071821	0.45018906	0.39501224	0.34681657	0.30469137	0.26784832	0.20736795
28	0.46785227	0.43707675	0.38165434	0.33347747	0.29157069	0.25509364	0.19563014
29	0.45533068	0.42434636	0.36874815	0.32065141	0.27901502	0.24294632	0.18455674

30	0.44314421	0.41198676	0.35627841	0.30831867	0.26700002	0.23137745	0.17411013
31	0.43128391	0.39998715	0.34423035	0.29646026	0.25550241	0.22035947	0.16425484
32	0.41974103	0.38833703	0.33258971	0.28505794	0.24449991	0.20986617	0.15495740
33	0.40850708	0.37702625	0.32134271	0.27409417	0.23397121	0.19987254	0.14618622
34	0.39757380	0.36604490	0.31047605	0.26355209	0.22389589	0.19035480	0.13791153
35	0.38693314	0.35538340	0.29997686	0.25341547	0.21425444	0.18129029	0.13010522
36	0.37657727	0.34503243	0.28983272	0.24366872	0.20502817	0.17265741	0.12274077
37	0.36649856	0.33498294	0.28003161	0.23429685	0.19619921	0.16443563	0.11579318
38	0.35668959	0.32522615	0.27056194	0.22528543	0.18775044	0.15660536	0.10923885
39	0.34714316	0.31575355	0.26141250	0.21662061	0.17966549	0.14914797	0.10305552
40	0.33785222	0.30655684	0.25257247	0.20828904	0.17192870	0.14204568	0.09722219
41	0.32880995	0.29762800	0.24403137	0.20027793	0.16452507	0.13528160	0.09171905
42	0.32000968	0.28895922	0.23577910	0.19257493	0.15744026	0.12883962	0.08652740
43	0.31144495	0.28054294	0.22780590	0.18516820	0.15066054	0.12270440	0.08162962
44	0.30310944	0.27237178	0.22010231	0.17804635	0.14417276	0.11686133	0.07700908
45	0.29499702	0.26443862	0.21265924	0.17119841	0.13796437	0.11129651	0.07265007
46	0.28710172	0.25673653	0.20546787	0.16461386	0.13202332	0.10599668	0.06853781
47	0.27941773	0.24925876	0.19851968	0.15828256	0.12633810	0.10094921	0.06465831
48	0.27193940	0.24199880	0.19180645	0.15219476	0.12089771	0.09614211	0.06099840
49	0.26466122	0.23495029	0.18532024	0.14634112	0.11569158	0.09156391	0.05754566
50	0.25757783	0.22810708	0.17905337	0.14071262	0.11070965	0.08720373	0.05428836
55	0.22490511	0.19676717	0.15075814	0.11565551	0.08883907	0.06832640	0.04056742
60	0.19637679	0.16973309	0.12693431	0.09506040	0.07128901	0.05353552	0.03031434
65	0.17146718	0.14641325	0.10687528	0.07813272	0.05720594	0.04194648	0.02265264
70	0.14971726	0.12629736	0.08998612	0.06421940	0.04590497	0.03286617	0.01692737
75	0.13072622	0.10894521	0.07576590	0.05278367	0.03683649	0.02575150	0.01264911
80	0.11414412	0.09397710	0.06379285	0.04338433	0.02955948	0.02017698	0.00945215
85	0.09966540	0.08106547	0.05371187	0.03565875	0.02372003	0.01580919	0.00706320
90	0.08702324	0.06992779	0.04522395	0.02930890	0.01903417	0.01238691	0.00527803
95	0.07598469	0.06032032	0.03807735	0.02408978	0.01527399	0.00970547	0.00394405
100	0.06634634	0.05203284	0.03206011	0.01980004	0.01225663	0.00760449	0.00294723

Table III—Part 1

Amount of Annuity of $1 at End of Each Period

Periods	1%	1 1/4%	1 1/2%	1 3/4%	2%	2 1/4%	2 1/2%
1	1.	1.	1.	1.	1.	1.	1.
2	2.01	2.0125	2.015	2.0175	2.02	2.0225	2.025
3	3.0301	3.03765625	3.045225	3.05280625	3.0604	3.06800625	3.075625
4	4.060401	4.07562695	4.09090338	4.10623036	4.121608	4.13703639	4.15251563
5	5.10100501	5.12657229	5.15226693	5.17808939	5.20404016	5.23011971	5.25632852
6	6.15201506	6.19065444	6.22955093	6.26870596	6.30812096	6.34779740	6.38773673
7	7.21353521	7.26803762	7.32299419	7.37840831	7.43428338	7.49062284	7.54743015
8	8.28567056	8.35888809	8.43283911	8.50753045	8.58296905	8.65916186	8.73611590
9	9.36852727	9.46337420	9.55933169	9.65641224	9.75462843	9.85399300	9.95451880
10	10.46221254	10.58166637	10.70272167	10.82539945	10.94972100	11.07570784	11.20338177
11	11.56683467	11.71393720	11.86326249	12.01484394	12.16871542	12.32491127	12.48346631
12	12.68250301	12.86036142	13.04121143	13.22510371	13.41208973	13.60222177	13.79555297
13	13.80932804	14.02111594	14.23682960	14.45654303	14.68033152	14.90827176	15.14044179
14	14.94742132	15.19637988	15.45038205	15.70953253	15.97393815	16.24370788	16.51895284
15	16.09689554	16.38633463	16.68213778	16.98444935	17.29341692	17.60919130	17.93192666
16	17.25786449	17.59116382	17.93236984	18.28167721	18.63928525	19.0053981	19.38022483
17	18.43044314	18.81105336	19.20135539	19.60160656	20.01207096	20.43301957	20.86473045
18	19.61474757	20.04619153	20.48937572	20.94463468	21.41231238	21.89276251	22.38634871
19	20.81089504	21.29676893	21.79671636	22.31116578	22.84055863	23.38534966	23.94600743
20	22.01900399	22.56297854	23.12366710	23.70161119	24.29736980	24.91152003	25.54465761
21	23.23919403	23.84501577	24.47052211	25.11638938	25.78331719	26.47202923	27.18327405
22	24.47158598	25.14307847	25.83757994	26.55592620	27.29898354	28.06764989	28.86285590
23	25.71630183	26.45736695	27.22514364	28.02065490	28.84496321	29.69917201	30.58442730
24	26.97346485	27.78808403	28.63352080	29.51101637	30.42186247	31.36740338	32.34903798
25	28.24319950	29.13543508	30.06302361	31.02745915	32.03029972	33.07316996	34.15776393
26	29.52563150	30.49962802	31.51396896	32.57043969	33.67090572	34.81731628	36.01170803
27	30.82088781	31.88087337	32.98667850	34.14042238	35.34432383	36.60070590	37.91200073
28	32.12909669	33.27938429	34.48147867	35.73787977	37.05121031	38.42422178	39.85980075
29	33.45038766	34.69537659	35.99870085	37.36329267	38.79223451	40.28876677	41.85629577

30	34.78489153	36.12906880	37.53868137	39.01715029	40.56807921	42.19526402	43.90270316
31	36.13274045	37.58068216	39.10176159	40.69995042	42.37944079	44.14465746	46.00027074
32	37.49406785	39.05044069	40.68828801	42.41219955	44.22702961	46.13791226	48.15027751
33	38.86900853	40.53857120	42.29861233	44.15441305	46.11157020	48.17601528	50.35403445
34	40.25769862	42.04530334	43.93309152	45.92711527	48.03380160	50.25997563	52.61288531
35	41.66027560	43.57086963	45.59208789	47.73083979	49.99447763	52.39082508	54.92820744
36	43.07687836	45.11550550	47.27596921	49.56612949	51.99436719	54.56961864	57.30141263
37	44.50764714	46.67944932	48.98510874	51.43353675	54.03425453	56.79743506	59.73394794
38	45.95272361	48.26294243	50.71988538	53.33362365	56.11493962	59.07537735	62.22729664
39	47.41225085	49.86622921	52.48068366	55.26696206	58.23723841	61.40457334	64.78297906
40	48.88637336	51.48955708	54.26789391	57.23413390	60.40198318	63.78617624	67.40255354
41	50.37523709	53.13317654	56.08191232	59.23573124	62.61002284	66.22136521	70.08761737
42	51.87898946	54.79734125	57.92314100	61.27235654	64.86222330	68.71134592	72.83980781
43	53.39777936	56.48230801	59.79198812	63.34462278	67.15946777	71.25735121	75.66080300
44	54.93175715	58.18833687	61.68886794	65.45315367	69.50265712	73.86064161	78.55232308
45	56.48107472	59.91569108	63.61420096	67.59858386	71.89271027	76.52250605	81.51613116
46	58.04588547	61.66463721	65.56841398	69.78155908	74.33056447	79.24426243	84.55403443
47	59.62634432	63.43544518	67.55194018	72.00273637	76.81717576	82.02725834	87.66788530
48	61.22260777	65.22838824	69.56521929	74.26278425	79.35351927	84.87287165	90.85958243
49	62.83483385	67.04374310	71.60869758	76.56238298	81.94058966	87.78251126	94.13107199
50	64.46318218	68.88178989	73.68282804	78.90222468	84.57940145	90.75761776	97.48434879
55	72.85245735	78.42245562	84.52959893	91.23016259	98.58653305	106.66788460	115.55092136
60	81.66966986	88.57450776	96.21465171	104.67521588	114.05153942	124.45043493	135.99158995
65	90.93664882	99.37712526	108.80277215	119.33861370	131.12615541	144.32559477	159.11833027
70	100.67633684	110.87199776	122.36375295	135.33075826	149.97791114	166.53961758	185.28411421
75	110.91284684	123.10348644	136.97278063	152.77205601	170.79177276	191.36773536	214.88829705
80	121.67152172	136.11879526	152.71085247	171.79382424	193.77195780	219.11756877	248.38271265
85	132.97899715	149.96815310	169.66522551	192.53927976	219.14393897	250.13293857	286.27856955
90	144.86326746	164.70500762	187.92990038	215.16461718	247.15665632	284.79812555	329.15425328
95	157.35375501	180.38623151	207.60614246	239.84018495	278.08495978	323.54263177	377.66415398
100	170.48138294	197.07234200	228.80304330	266.75176789	312.23230591	366.84650213	432.54865404

Table III—Part 2

Amount of Annuity of $1 at End of Each Period

Periods	2 3/4%	3%	3 1/2%	4%	4 1/2%	5%	6%
1	1.	1.	1.	1.	1.	1.	1.
2	2.0275	2.03	2.035	2.04	2.045	2.05	2.06
3	3.08325625	3.0909	3.106225	3.1216	3.137025	3.1525	3.1836
4	4.16804580	4.183627	4.21494288	4.246464	4.27819113	4.310125	4.374616
5	5.28266706	5.30913581	5.36246588	5.41632256	5.47070973	5.52563125	5.63709296
6	6.42794040	6.46840988	6.55015218	6.63297546	6.71689166	6.80191281	6.97531854
7	7.60470876	7.66246218	7.77940751	7.89829448	8.01915179	8.14200845	8.39383765
8	8.81383825	8.89233605	9.05168677	9.21422626	9.38001362	9.54910888	9.89746791
9	10.05621880	10.15910613	10.36849581	10.58279531	10.80211423	11.02656432	11.49131598
10	11.33276482	11.46387931	11.73139316	12.00610712	12.28820937	12.57789254	13.18079494
11	12.64441585	12.80779569	13.14199192	13.48635141	13.84117879	14.20678716	14.97164264
12	13.99213729	14.19202956	14.60196164	15.02580546	15.46403184	15.91712652	16.86994120
13	15.37692107	15.61779045	16.11303030	16.62683768	17.15991327	17.71298285	18.88213767
14	16.79978639	17.08632416	17.67698636	18.29191119	18.93210937	19.59863199	21.01506593
15	18.26178052	18.59891389	19.29568088	20.02358764	20.78405429	21.57856359	23.27596988
16	19.76397948	20.15688130	20.97102971	21.82453114	22.71933673	23.65749177	25.67252808
17	21.30748892	21.76158774	22.70501575	23.69751239	24.74170689	25.84036636	28.21287976
18	22.89344487	23.41443537	24.49969130	25.64541288	26.85508370	28.13238467	30.90565255
19	24.52301460	25.11686844	26.35718050	27.67122940	29.06356246	30.53900391	33.75999170
20	26.19739750	26.87037449	28.27968181	29.77807858	31.37142277	33.06595410	36.78559120
21	27.91782593	28.67648572	30.26947068	31.96920172	33.78313680	35.71925181	39.99272668
22	29.68556615	30.53678030	32.32890215	34.24796979	36.30337795	38.50521440	43.39229028
23	31.50191921	32.45288370	34.46041373	36.61788858	38.93702996	41.43047512	46.99582769
24	33.36822199	34.42647022	36.66652821	39.08260412	41.68919631	44.50199887	50.81557735
25	35.28584810	36.45926432	38.94985669	41.64590829	44.56521015	47.72709882	54.86451200
26	37.25620892	38.55304225	41.31310168	44.31174462	47.57064460	51.11345376	59.15638272
27	39.28075467	40.70963352	43.75906024	47.08421440	50.71132361	54.66912645	63.70576568
28	41.36097542	42.93092252	46.29062734	49.96758298	53.99333317	58.40258277	68.52811162
29	43.49840224	45.21885020	48.91079930	52.96628630	57.42303316	62.32271191	73.63979832

30	45.69460830	47.57541571	51.62267728	56.08493775	61.00706966	66.43884750	79.05818622
31	47.95121003	50.00267818	54.42947098	59.32833526	64.75238779	70.76078988	84.80167739
32	50.26986831	52.50275852	57.33450247	62.70146867	68.66624524	75.29882937	90.88977803
33	52.65228969	55.07784128	60.34121005	66.20952742	72.75622628	80.06377084	97.34316471
34	55.10022765	57.73017652	63.45315240	69.85790851	77.03025646	85.06695938	104.18375460
35	57.61548391	60.46208181	66.67401274	73.65222486	81.49661800	90.32030735	111.43477987
36	60.19990972	63.27594427	70.00760318	77.59831385	86.16396581	95.83632272	119.12086666
37	62.85540724	66.17422259	73.45786930	81.70224640	91.04134427	101.62813886	127.26811866
38	65.58393094	69.15944927	77.02889472	85.97033626	96.13820476	107.70954580	135.90420578
39	68.38748904	72.23423275	80.72490604	90.40914971	101.46442398	114.09502309	145.05845813
40	71.26814499	75.40125973	84.55027775	95.02551570	107.03032306	120.79977424	154.76196562
41	74.22801898	78.66329753	88.50953747	99.82653633	112.84668760	127.83976295	165.04768356
42	77.26928950	82.02319645	92.60737128	104.81959778	118.92478854	135.23175110	175.95054457
43	80.39419496	85.48389234	96.84862928	110.01238169	125.27640402	142.99333866	187.50757724
44	83.60503532	89.04840911	101.23833130	115.41287696	131.91384220	151.14300559	199.75803188
45	86.90417379	92.71986139	105.78167290	121.02939204	138.84996510	159.70015587	212.74351379
46	90.29403857	96.50145723	110.48403145	126.87056772	146.09821353	168.68516366	226.50812462
47	93.77712463	100.39650095	115.35097255	132.94539043	153.67263314	178.11942185	241.09861210
48	97.35599556	104.40839598	120.38825659	139.26320604	161.58790163	188.02539294	256.56452882
49	101.03328544	108.54064785	125.60184557	145.83373429	169.85935720	198.42666259	272.95840055
50	104.81170079	112.79686729	130.99791016	152.66708366	178.50302828	209.34799572	290.33590458
55	125.32071411	136.07161972	160.94688984	191.15917299	227.91795938	272.71261833	394.17202657
60	148.80914038	163.05343680	196.51688288	237.99068520	289.49795398	353.58371788	533.12818089
65	175.70980889	194.33275782	238.76287650	294.96838045	366.23783096	456.79801118	719.08286076
70	206.51842746	230.59406374	288.93786459	364.29045876	461.86967955	588.52851071	967.93216965
75	241.80271709	272.63085559	348.53001083	448.63136652	581.04436193	756.65371848	1300.94867977
80	282.21287345	321.36301855	419.30678685	551.24497675	729.55769854	971.22882134	1746.59989137
85	328.49354837	377.85695165	503.36739448	676.09012345	914.63233612	1245.08706889	2342.98174142
90	381.49757170	443.34890365	603.20502701	827.98333354	1145.26900659	1594.60730098	3141.07518718
95	442.20166674	519.27202569	721.78081595	1012.78464845	1432.68425949	2040.69352892	4209.10424961
100	511.72444867	607.28773270	862.61165666	1237.62370461	1790.85595627	2610.02515693	5638.36805857

TABLE IV—PART 1

PRESENT WORTH OF ANNUITY OF $1 AT END OF EACH PERIOD

PERIODS	1%	1 1/4%	1 1/2%	1 3/4%	2%	2 1/4%	2 1/2%
1	0.99009901	0.98765432	0.98522167	0.98280098	0.98039216	0.97799511	0.97560976
2	1.97039506	1.96311538	1.95588342	1.94869875	1.94156094	1.93446955	1.92742415
3	2.94098521	2.92653371	2.91220042	2.89798403	2.88388327	2.86989687	2.85602356
4	3.90196555	3.87805798	3.85438465	3.83094254	3.80772870	3.78474021	3.76197421
5	4.85343124	4.81783504	4.78264497	4.74785508	4.71345951	4.67945253	4.64582850
6	5.79547647	5.74600992	5.69718717	5.64899762	5.60143089	5.55447680	5.50812536
7	6.72819453	6.66272585	6.59821396	6.53464139	6.47199107	6.41024626	6.34939060
8	7.65167775	7.56812429	7.48592508	7.40505297	7.32548144	7.24718461	7.17013717
9	8.56601758	8.46234498	8.36051732	8.26049432	8.16223671	8.06570622	7.97086553
10	9.47130453	9.34552591	9.22218455	9.10122291	8.98258501	8.86621635	8.75206393
11	10.36762825	10.21780337	10.07111779	9.92749181	9.78684805	9.64911134	9.51420871
12	11.25507747	11.07931197	10.90750521	10.73954969	10.57534122	10.41477882	10.25776460
13	12.13374007	11.93018466	11.73153222	11.53764097	11.34837375	11.16359787	10.98318497
14	13.00370304	12.77055275	12.54338150	12.32200587	12.10624877	11.89593924	11.69091217
15	13.86505252	13.60054592	13.34323301	13.09288046	12.84926350	12.61216551	12.38137773
16	14.71787378	14.42029227	14.13126405	13.85049677	13.57770931	13.31263131	13.05500266
17	15.56225127	15.22991829	14.90764931	14.59508282	14.29187188	13.99768343	13.71219772
18	16.39826858	16.02954893	15.67256089	15.32686272	14.99203125	14.66766106	14.35336363
19	17.22600850	16.81930759	16.42616837	16.04605673	15.67846201	15.32289590	14.97889134
20	18.04555297	17.59931613	17.16863879	16.75288130	16.35143334	15.96371237	15.58916229
21	18.85698313	18.36969495	17.90013673	17.44754919	17.01120916	16.59042775	16.18454857
22	19.66037934	19.13056291	18.62082437	18.13026948	17.65804820	17.20335232	16.76541324
23	20.45582113	19.88203744	19.33086145	18.80124764	18.29220412	17.80278955	17.33211048
24	21.24338726	20.62423451	20.03040537	19.46068565	18.91392560	18.38903624	17.88498583
25	22.02315570	21.35726865	20.71961120	20.10878196	19.52345647	18.96238263	18.42437642
26	22.79520366	22.08125299	21.39863172	20.74573166	20.12103576	19.52311260	18.95061114
27	23.55960759	22.79629925	22.06761746	21.37172644	20.70689780	20.07150376	19.46401087
28	24.31644316	23.50251778	22.72671671	21.98695474	21.28127236	20.60782764	19.96488866
29	25.06578530	24.20001756	23.37607558	22.59160171	21.84438466	21.13234977	20.45354991

30	25.80770822	24.88890623	24.01583801	23.18584934	22.39645555	21.64532985	20.93029259
31	26.54228537	25.56929010	24.64614582	23.76987650	22.93770152	22.14702186	21.39540741
32	27.26958947	26.24127418	25.26713874	24.34385897	23.46833482	22.63767419	21.84917796
33	27.98969255	26.90496215	25.87895442	24.90796951	23.98856355	23.11752977	22.29188094
34	28.70266589	27.56045644	26.48172849	25.46237789	24.49859172	23.58682618	22.72378628
35	29.40858009	28.20785822	27.07559458	26.00725100	24.99861933	24.04579577	23.14515734
36	30.10750504	28.84726737	27.66068431	26.54275283	25.48884248	24.49466579	23.55625107
37	30.79950994	29.47878259	28.23712740	27.06904455	25.96945341	24.93365848	23.95731812
38	31.48466330	30.10250133	28.80505163	27.58628457	26.44064060	25.36299118	24.34860304
39	32.16303298	30.71851983	29.36458288	28.09462857	26.90258883	25.78287646	24.73034443
40	32.83468611	31.32693316	29.91584520	28.59422955	27.35547924	26.19352221	25.10277505
41	33.49968922	31.92783522	30.45896079	29.08523789	27.79948945	26.59513174	25.46612200
42	34.15810814	32.52131874	30.99405004	29.56780136	28.23479358	26.98790390	25.82060683
43	34.81000806	33.10747530	31.52123157	30.04206522	28.66156233	27.37203316	26.16644569
44	35.45545352	33.68639536	32.04062223	30.50817221	29.07996307	27.74770969	26.50384945
45	36.09450844	34.25816825	32.55233718	30.96626261	29.49015987	28.11511950	26.83302386
46	36.72723608	34.82288222	33.05648983	31.41647431	29.89231360	28.47444450	27.15416962
47	37.35369909	35.38062442	33.55319195	31.85894281	30.28658196	28.82586259	27.46748255
48	37.97395949	35.93148091	34.04255363	32.29380129	30.67311957	29.16954777	27.77315371
49	38.58807871	36.47553670	34.52468339	32.72118063	31.05207801	29.50567019	28.07136947
50	39.19611753	37.01287574	34.99968807	33.14120946	31.42360589	29.83439627	28.36231168
55	42.14719216	39.60168667	37.27146681	35.13544550	33.17478752	31.37265438	29.71397928
60	44.95503841	42.03459179	39.38026889	36.96398552	34.76088668	32.74895285	30.90865649
65	47.62660777	44.32098022	41.33778618	38.64059678	36.19746555	33.98034405	31.96457705
70	50.16851435	46.46967562	43.15487183	40.17790267	37.49861929	35.08208492	32.89785698
75	52.58705124	48.48897027	44.84160034	41.58747771	38.67711433	36.06782605	33.72274044
80	54.88820611	50.38665706	46.40732349	42.87993474	39.74451359	36.94978079	34.45181722
85	57.07767600	52.17005958	47.86072218	44.06500479	40.71128999	37.73887655	35.09621486
90	59.16088148	53.84606035	49.20985452	45.15161037	41.58692916	38.44489025	35.66576848
95	61.14298002	55.42112744	50.46220054	46.14793265	42.38002254	39.07656940	36.16917089
100	63.02887877	56.90133936	51.62470367	47.06147304	43.09835164	39.64174052	36.61410526

TABLE IV—PART 2

PRESENT WORTH OF ANNUITY OF $1 AT END OF EACH PERIOD

PERIODS	2 3/4%	3%	3 1/2%	4%	4 1/2%	5%	6%
1	0.97323601	0.97087379	0.96618357	0.96153846	0.95693780	0.95238095	0.94339623
2	1.92042434	1.91346970	1.89969428	1.88609467	1.87266775	1.85941043	1.83339267
3	2.84226213	2.82861135	2.80163698	2.77509103	2.74896435	2.72324803	2.67301195
4	3.73942787	3.71709840	3.67307921	3.62989522	3.58752570	3.54595050	3.46510561
5	4.61258186	4.57970719	4.51505238	4.45182233	4.38997674	4.32947667	4.21236379
6	5.46236678	5.41719144	5.32855302	5.24213686	5.15787248	5.07569206	4.91732433
7	6.28940806	6.23028296	6.11454398	6.00205467	5.89270094	5.78637340	5.58238144
8	7.09431441	7.01969219	6.87395554	6.73274487	6.59588607	6.46321276	6.20979381
9	7.87767826	7.78610892	7.60768651	7.43533161	7.26879050	7.10782168	6.80169227
10	8.64007616	8.53020284	8.31660532	8.11089578	7.91271818	7.72173493	7.36008705
11	9.38206926	9.25262411	9.00155104	8.76047671	8.52891692	8.30641422	7.88687458
12	10.10420366	9.95400399	9.66333433	9.38507376	9.11858078	8.86325164	8.38384394
13	10.80701086	10.63495533	10.30273849	9.98564785	9.68285242	9.39357299	8.85268296
14	11.49100814	11.29607314	10.92052028	10.56312293	10.22282528	9.89864094	9.29498393
15	12.15669892	11.93793509	11.51741090	11.11838743	10.73954573	10.37965804	9.71224899
16	12.80457315	12.56110203	12.09411681	11.65229561	11.23401505	10.83776956	10.10589527
17	13.43510769	13.16611847	12.65132059	12.16566885	11.70719143	11.27406625	10.47725969
18	14.04876661	13.75351308	13.18968173	12.65929697	12.15999180	11.68958690	10.82760348
19	14.64600157	14.32379911	13.70983742	13.13393940	12.59329359	12.08532086	11.15811649
20	15.22725213	14.87747486	14.21240330	13.59032634	13.00793645	12.46221034	11.46992122
21	15.79294612	15.41502414	14.69797420	14.02915995	13.40472388	12.82115271	11.76407662
22	16.34349987	15.93691664	15.16712484	14.45111533	13.78442476	13.16300258	12.04158172
23	16.87931861	16.44360839	15.62041047	14.85684167	14.14777489	13.48857388	12.30337898
24	17.40079670	16.93554212	16.05836760	15.24696314	14.49547837	13.79864179	12.55035753
25	17.90831795	17.41314769	16.48151459	15.62207994	14.82820896	14.09394457	12.78335616
26	18.40225592	17.87684242	16.89035226	15.98276918	15.14661145	14.37518530	13.00316619
27	18.88297413	18.32703147	17.28536451	16.32958575	15.45130282	14.64303362	13.21053414
28	19.35082640	18.76410823	17.66701885	16.66306322	15.74287351	14.89812726	13.40616428
29	19.80615708	19.18845459	18.03576700	16.98371463	16.02188853	15.14107358	13.59072102

30	20.24930130	19.60044135	18.39204541	17.29203330	16.28888854	15.37245103	13.76483115
31	20.68058520	20.00042849	18.73627576	17.58849356	16.54439095	15.59281050	13.92908599
32	21.10032623	20.38876553	19.06886547	17.87355150	16.78889086	15.80267667	14.08404339
33	21.50883332	20.76579178	19.39020818	18.14764567	17.02286207	16.00254921	14.23022961
34	21.90640712	21.13183668	19.70068423	18.41119776	17.24675796	16.19290401	14.36814114
35	22.29334026	21.48722007	20.00066110	18.66461323	17.46101240	16.37419429	14.49824636
36	22.66991753	21.83225250	20.29049381	18.90828195	17.66604058	16.54685171	14.62098713
37	23.03641609	22.16723544	20.57052542	19.14257880	17.86223979	16.71128734	14.73678031
38	23.39310568	22.49246159	20.84108736	19.36786423	18.04999023	16.86789271	14.84601916
39	23.74024884	22.80821513	21.10249987	19.58448484	18.22965572	17.01704067	14.94907468
40	24.07810106	23.11477197	21.35507234	19.79277388	18.40158442	17.15908635	15.04629687
41	24.40691101	23.41239997	21.59910371	19.99305181	18.56610949	17.29436796	15.13801592
42	24.72692069	23.70135920	21.83488281	20.18562674	18.72354975	17.42320758	15.22454332
43	25.03836563	23.98190213	22.06268870	20.37079494	18.87421029	17.54591198	15.30617294
44	25.34147507	24.25427392	22.28279102	20.54884129	19.01838305	17.66277331	15.38318202
45	25.63647209	24.51871254	22.49545026	20.72003970	19.15634742	17.77406982	15.45583209
46	25.92357381	24.77544907	22.70091813	20.88465356	19.28837074	17.88006650	15.52436990
47	26.20299154	25.02470783	22.89943780	21.04293612	19.41470884	17.98101571	15.58902821
48	26.47493094	25.26670664	23.09124425	21.19513088	19.53560654	18.07715782	15.65002661
49	26.73959215	25.50165693	23.27656450	21.34147200	19.65129813	18.16872173	15.70757227
50	26.99716998	25.72976401	23.45561787	21.48218462	19.76200778	18.25592546	15.76186064
55	28.18526879	26.77442764	24.26405323	22.10861218	20.24802057	18.63347196	15.99054297
60	29.22266201	27.67556367	24.94473412	22.62348997	20.63802204	18.92928952	16.16142771
65	30.12846605	28.45289152	25.51784916	23.04668199	20.95097913	19.16107033	16.28912272
70	30.91937247	29.12342135	26.00039664	23.39451498	21.20211187	19.34267665	16.38454387
75	31.60995558	29.70182628	26.40668868	23.68040834	21.40363360	19.48496995	16.45584810
80	32.21294098	30.20076345	26.74877567	23.91539185	21.56534493	19.59646048	16.50913077
85	32.73944009	30.63115103	27.03680373	24.10853116	21.69511035	19.68381623	16.54894668
90	33.19915489	31.00240714	27.27931564	24.26727759	21.79924075	19.75226174	16.57869944
95	33.60055671	31.32265592	27.48350415	24.39775559	21.88280030	19.80589059	16.60093244
100	33.95104232	31.59890534	27.65542540	24.50499900	21.94985274	19.84791020	16.61754623

INDEX

A

Accounts, averaging, 42–49
Addition,
  fractions, 16
  short methods, 4
Amortization, bonds, 194–216
Annuities, 171–189
  logarithms, 190–193
  present worth, 186
  tables, 262–269
Arithmetical progression, 22–28
Averages, 29–41
  accounts, 42–49
  compound, 47–49
  dates, 42–45
    in foreign exchange, 128
  interest, 42
  moving, 30
  periodic, 37
  progressive, 34
  settling an account, 42
  simple, 29
  weighted, 38

B

Balancing, short methods, 3
Banking, clearing house problems, 98–102
Bonds,
  amortization, 194–216
  discount, 194–216
  income rates, 201
  premiums, 203
  present worth, 205
  price computation, 205
Building and loan associations, 103–115
  Dayton or Ohio plan, 112
  Dexter's rule, 108
  income of, 110
  individual plan, 111
  partnership plan, 107
  premiums, 110
  serial plan, 106
  terminating plan, 103
  withdrawal of shares, 110

C

Capital,
  capitalizing gross income, 118
  reducing to profit and loss ratio, 95
  working,
    basis of turnover, 85
    defined, 87
Cash discount, 80–82
Clearing house, 98–102
Complements, 8
Consolidations, 116–122
  ascertaining earning power, 116
  capitalizing gross income, 118
  stock issues, 116, 120
  valuation of good-will, 117
Conversion,
  foreign branch accounts, 130
  foreign exchange, 123–134
Currency, foreign, table of values, 231

D

Dates,
  averages, 42–45, 128

Dates—*Continued*
current account in foreign exchange, 128
reducing to months, 45
Dayton plan, building and loan associations, 112
Debts, present value, 182
Decrease (See "Increase and decrease")
Depreciation, 221–230
Dexter's rule, 108
Discount (See also "Amortization")
cash, 80–82
compound, 169
trade, 76–79
Division,
logarithms, 137
short methods, 13

E

Earnings, ascertaining of, 116
Equations, 66–75
Exchange (See "Foreign exchange")

F

Focal date, 44
Foreign branches, conversion of accounts, 130
Foreign exchange, 123–134
average date of current account, 128
table of values, 231
Fractions, 15–20

G

Gross profit method approximating inventory, 56
Good-will, 116–122
valuation of, 117

I

Income,
capitalizing gross, 118
consolidations, 116–122
Increase and decrease,
averages, 32–37
percentage, 51
short methods, 7
Individual plan, building and loan association, 111
Interest, 42, 154–170
logarithms, 190–193
rents, 217–220
tables, 254–269
Interpolation, logarithms, 150
Inventory,
gross profit method of approximating, 56
turnover, 84

L

Leaseholds, 217–220
Liquidation, of partnership, 94
Logarithms, 135–153
annuities, 190–193
characteristic, 140
negative, 146
compound interest, 190–193
division, 137
interpolation, 150
mantissa, 140
multiplication, 136
nature of, 140
powers, 137
roots, 139
tables, 233–253

M

Mantissa, 140
Multiplication,
fractions, 16
logarithms, 136
short methods, 9–13

N

Net decrease, short method, 7

O

Ohio plan, building and loan associations, 112

P

Partial payments,
building and loan associations, 103
interest calculations, 157
Partnership, 89–97
building and loan associations, 197
liquidation, 94
periodical distribution, 94
profit and loss, 89–94
Percentage, 50–65
apportionment, 55
inventory, 56
sales, 51–55
statements, analysis of, 60–65
Powers, logarithms, 137
Premiums, building and loan associations, 110
Present worth,
annuities, 175
annuities by logarithms, 190–193
bonds, 205
calculation of, 167
debts, 182
rent, 179
sinking funds, 174, 185
tables, 254–269
Price, trade discount, 77
Profit and loss,
partnership, 89–94
reducing capital to ratio, 95
Progressions, arithmetical, 22–28
Proofs, 20

R

Reciprocals, division by use of, 13
Rent, 217–220
Rent—*Continued*
present worth, 179
Roots, logarithms, 139

S

Sales, percentage, 51–55
Serial plan, building and loan associations, 106
Shares (See "Stock")
Short methods,
addition, 4
balancing, 3
division, 13
increase and decrease, 7
multiplication, 9–13
subtraction, 6
Sinking funds, present worth, 174, 185
Stock,
building and loan associations, methods of, 103–115
issues of, in consolidation, 116, 120
Stock-in-trade (See "Inventory")
Solutions, equations, 66–75
Statements, percentage analysis, 60–65
Subtraction,
fractions, 16
short methods, 6

T

Tables,
foreign currency, 231
interest, 254–269
logarithms, 233–253
Terminal plan, building and loan associations, 103
Trade discount, 76–79
Turnover, 83–88
working capital as basis, 85

V

Valuation,
  good-will, 117
Values,
  foreign currency, 231

W

Weighted averages, 38
Withdrawal of shares,
  Building and Loan Associations, 110
Working capital,
  basis of turnover, 85
  defined, 87
Worth,
  present (See " Present worth ")

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